I. E. OVCHARENKO

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Abstract

Full Text

I. E. OVCHARENKO

ON ONE APPLICATION OF THE METHOD OF DIRECTING FUNCTIONALS IN THE THEORY OF PRECOMMUTING OPERATORS

(Presented by Academician L. S. Pontryagin, 21 X 1963)

1. Let A be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(A)$ dense in it, and let B be a bounded operator defined everywhere in $\mathfrak{H}$. We shall say that the operators A and B **precommute** if, for any $f, g \in \mathfrak{D}(A)$, the equality*

$$(Bf, Ag) = (Af, B^*g) \tag{1}$$

holds.

Let $\widetilde{\mathfrak{H}}$ ($\widetilde{\mathfrak{H}} \supset \mathfrak{H}$) be a certain Hilbert space and let B^+ be a bounded operator acting in it. We shall call B^+ a block extension of the operator B if $PB^+f = Bf$ ($f \in \mathfrak{H}$), where P is the operator of orthogonal projection of $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}$.

Theorem 1. *Let a closed symmetric operator A in a Hilbert space $\mathfrak{H}$, possessing a finite system of directing functionals**, precommute with a bounded operator B. Then there exist a self-adjoint extension $\widetilde{A}$ of the operator A, acting in a space $\widetilde{\mathfrak{H}}$ ($\widetilde{\mathfrak{H}} \supset \mathfrak{H}$), and a block extension B^+ of the operator B with the same norm as B, such that $\widetilde{A}$ and B commute with each other.*

Proof. We carry out the argument under the assumption that the operator A has one directing functional. The argument in the general case is analogous. Without loss of generality one may assume that $\|B\| = 1$. Form the set of pairs $\{\varphi_j\}_1^2$, $\varphi_j \in \mathfrak{H}$. Algebraic operations are introduced in the natural way, and the quasi-scalar product is defined by the equality

$$(\{\varphi_j\}, \{\psi_j\})_1 = (\varphi_1, \psi_1) + (\varphi_2, \psi_2) + (B\varphi_2, \psi_1) + (B^*\varphi_1, \psi_2). \tag{2}$$

In the resulting quasi-Hilbert space we define the operator $A' = A \dot{+} A$. From (1) it follows that A' is symmetric. If the functional $\Phi(f; \lambda)$ is directing for A, then the system of functionals $\Phi(\varphi_j; \lambda)$, $j = 1, 2$, is directing for the operator A'. By virtue of the fundamental proposition on operators with directing functionals

(1), there exists a nondecreasing matrix-function $T_1(\lambda) = \|\sigma_{jk}(\lambda)\|$ such that the equality

$$(\{\varphi_j\}, \{\psi_j\})_1 = \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \Phi(\varphi_j; \lambda)\, \overline{\Phi(\psi_k; \lambda)}\, d\sigma_{jk}(\lambda) \tag{3}$$

will hold.

Form the matrix-function $T(\lambda) = \sigma_{\|j-k\|}(\lambda)$ $(j, k = 1, 2)$, where $\sigma_0(\lambda) = \frac{1}{2}[\sigma_{11}(\lambda) + \sigma_{22}(\lambda)]$, $\sigma_1(\lambda) = \sigma_{-1}(\lambda) = \sigma_{12}(\lambda)$. From equalities (2) and (3) it follows that, for $\varphi \in \mathfrak{H}$,

$$\int_{-\infty}^{\infty} |\Phi(\varphi; \lambda)|^2\, d\sigma_{11}(\lambda) = \int_{-\infty}^{\infty} |\Phi(\varphi; \lambda)|^2\, d\sigma_{22}(\lambda) = (\varphi, \varphi).$$

* It is easy to see that if A is self-adjoint, then precommutation coincides with commutation in the usual sense.

** In what follows we adhere to the terminology and notation of the papers (1-3), in which an exposition is given of M. G. Krein’ s method of directing functionals.

Therefore one may regard $\mathfrak{H}$ as isometrically embedded in the space $\mathcal{L}_\sigma^{(2)}$. The operator $\widetilde{A}$ of multiplication by λ in the space $\mathcal{L}_\sigma^{(2)}$ is a self-adjoint extension of the operator A. From the monotonicity of $T_1(\lambda)$, and hence also of $T(\lambda)$, it follows that $|\Delta\sigma_1| \leq |\Delta\sigma_0|$. In view of the latter, on the space $\mathcal{L}_\sigma^{(2)}$ one can define the bounded bilinear functional

$$(f, g)_+ = \int_{-\infty}^{\infty} f(\lambda)\, \overline{g(\lambda)}\, d\sigma_1(\lambda),$$

and consequently also the bounded operator B^+, putting $(B^+ f, g) = (f, g)_+$. Comparison of equalities (2) and (3) shows that B^+ is a block extension of B, and the inequality $|\Delta\sigma_1| \leq |\Delta\sigma_0|$ shows that $\|B^+\| = \|B\|$. The commutation of $\widetilde{A}$ and B^+ is verified directly. The theorem is proved.

Let us note that, if the operator A has one directing functional, then the block extension B^+ constructed in the proof of Theorem 1 is a normal operator.

Up to unitary equivalence, all commuting pairs of extensions of the operators A and B that are mentioned in the theorem can be obtained by the method indicated in the proof. We shall explain this for the case when the operator A has one directing functional and in $\mathfrak{H}$ there exists a vector u biorthogonal to it, i.e. $\Phi(u; \lambda) \equiv 1$. As the spectral matrix of the operator A' one may take the matrix $\|\sigma_{j-k}(\lambda)\|$ $(j, k = 1, 2)$, where $\sigma_0(\lambda) = (E_\lambda u, u)$, $\sigma_1(\lambda) = \overline{\sigma_{-1}(\lambda)} =$

$(B^+\widetilde{E}_\lambda u, u)$, $\widetilde{E}_\lambda$ is the spectral family of the self-adjoint extension of the operator A commuting with the operator B^+, a block extension of the operator B. It is immediately clear that to different commuting pairs of extensions there correspond different spectral matrices of the operator A'. The latter circumstance, together with the known criterion for uniqueness of the spectral matrix of a symmetric operator [2], makes it possible to obtain the following proposition.

Theorem 2. *If, under the hypotheses of Theorem 1, there exist in* $\mathfrak{H}$ $4n$ *vectors* $\varphi_1, \varphi_2, \dots, \varphi_{2n}, \psi_1, \psi_2, \dots, \psi_{2n}$ *of unit length such that* $(B\varphi_k, \psi_k) = -\|B\|$ $(k = 1, 2, \dots, 2n)$ *and* $\det \|a_{ik}(\lambda)\| \not\equiv 0$*, where* $a_{ik}(\lambda) = \Phi_i(\varphi_k; \lambda)$*,* $i = 1, 2, \dots, n;$ $k = 1, 2, \dots, 2n$*,* $a_{ik}(\lambda) = \Phi_{i-n}(\psi_k; \lambda)$*,* $i = n+1, \dots, 2n;$ $k = 1, 2, \dots, 2n$*, then there is only one block extension of the operator* B *with the same norm as* B*, and only one self-adjoint extension of the operator* A*, which commute with each other.*

2. We shall apply the results obtained to one problem of the type of the moment problem. A function $F(t; j)$, $-2a \le t \le 2a;$ $j = 0, \pm 1, \dots, \pm n$, will be called Hermitian positive if the kernel $F(t-s; j-k)$ is positive definite. For continuous functions the latter is equivalent to the fact that, for every vector-function $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, whose coordinates are functions of bounded variation on the interval $(-a, a)$, the inequality

$$\sum_{j,k} \int_{-a}^{a} \int_{-a}^{a} F(t-s; j-k)\, d\varphi_j(t)\, \overline{d\varphi_k(s)} \ge 0$$

holds.

It is natural to pose the question of extending such a function while preserving Hermitian positivity for all values of t and all integral j.

Theorem 3. *Every continuous Hermitian positive function* $F(t; j)$*, given for* $-2a \le t \le 2a$*,* $j = 0, \pm 1$*, is representable in the form*

$$F(t; j) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\sigma_j(\lambda), \qquad j = 0,\ \pm 1, \tag{4}$$

where the matrix-function

$$T(\lambda) = \begin{pmatrix} \sigma_0(\lambda) & \sigma_1(\lambda) \\ \sigma_{-1}(\lambda) & \sigma_0(\lambda) \end{pmatrix}$$

is nondecreasing with bounded variation.

Let us outline the proof. Consider generalized functions of the form $f(x) = d\omega_f(x)/dx$, where $\omega_f(x)$ are functions of bounded variation normalized by the condition $\omega_f(-a) = 0$. On this set we define a quasi-scalar product by setting ([2,3])

$$(f,g)=\int_{-a}^{a}\int_{-a}^{a}F(t-s;0)\,d\omega_f(t)\,d\overline{\omega_g(s)}.$$

We define the operator $A = i\,d/dx$ on differentiable functions that vanish at the endpoints of the interval. A is symmetric and has one directing functional

$$\Phi(f;\lambda)=\int_{-a}^{a}e^{i\lambda s}\,d\omega_f(s).$$

The form

$$(f,g)_1=\int_{-a}^{a}\int_{-a}^{a}F(t-s;1)\,d\omega_f(t)\,d\overline{\omega_g(s)}$$

is representable in the form $(f,g)_1=(Bf,g)$, with $\|B\|\leqslant 1$. This follows from the inequality $|(f,g)_1|^2\leqslant(f,f)(g,g)$.

The precommutation of A and B is obvious. Let $\widetilde{A}$ and B^+ be commuting extensions of the operators A and B, whose existence was established in Theorem 1. It can be shown that

$$F(t;0)=(e^{it\widetilde{A}}u,u),\qquad u=\delta(x),\qquad -2a\leqslant t\leqslant 2a. \tag{5}$$

Recalling the definition of the operator B and taking into account that B^+ is a block extension of B commuting with $\widetilde{A}$, we have

$$F(t;1)=(B^+e^{it\widetilde{A}}u,u),\qquad -2a\leqslant t\leqslant 2a. \tag{6}$$

Denoting the spectral family of the operator $\widetilde{A}$ by $\widetilde{E}_\lambda$, we construct the distributions

$$\sigma_0(\lambda)=(\widetilde{E}_\lambda u,u),\qquad \sigma_1(\lambda)=(B^+\widetilde{E}_\lambda u,u),\qquad \sigma_{-1}(\lambda)=\overline{\sigma_1(\lambda)}.$$

In view of equalities (5) and (6), the function

$$\widetilde{F}(t;j)=\int_{-\infty}^{\infty}e^{i\lambda t}\,d\sigma_j(\lambda),\qquad -\infty<t<\infty,\qquad j=0,\pm1,$$

is an extension of the given function $F(t;j)$. Representation (4) is proved.

We note that, relying on Theorem 2, one can construct a Hermitian positive function $F(t;j)$, $-2a\leqslant t\leqslant 2a$, $j=0,\pm1$, which is uniquely extendable, whereas the function $F(t;0)$ may be any non-uniquely extendable Hermitian positive function.

Let the function $F(t;j)$, $-2a \leqslant t \leqslant 2a$, $j = 0, \pm 1$, be such that $F(t;0)$ is uniquely extendable from the interval $(-2a, 2a)$, or let $F(t;j)$ be given for $-\infty < t < \infty$, $j = 0, \pm 1$. The possibility of extending such a function in the discrete argument while preserving Hermitian positivity follows from a theorem of M. S. Livshits (see ([4])). We shall indicate a concrete method of extension. The operator A is self-adjoint,

$$B = \int_{-\infty}^{\infty} \varphi(\lambda)\, dE_\lambda,$$

where E_λ is the spec-

the central family A, $|\varphi(\lambda)| \leqslant 1$. We find the function $\varphi(\lambda)$ by the inversion formula from the equalities

$$F(t;1) = \int_{-\infty}^{\infty} e^{i\lambda t}\varphi(\lambda)\, d\sigma_0(\lambda), \qquad F(t;0) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\sigma_0(\lambda).$$

Define the continuation $F(t;j)$ by putting

$$\widetilde{F}(t;j) = \int_{-\infty}^{\infty} e^{i\lambda t}[\varphi(\lambda)]^j\, d\sigma_0(\lambda), \qquad j = 0, 1, 2, \dots;$$

$$\widetilde{F}(t;j) = \int_{-\infty}^{\infty} e^{i\lambda t}[\overline{\varphi(\lambda)}]^{-j}\, d\sigma_0(\lambda), \qquad j = -1, -2, \dots$$

Theorem 4. *A continuous Hermitian positive function $F(t;j)$, defined for $-2a \leqslant t \leqslant 2a$, $j = 0, \pm 1$, can always be continued in both arguments.*

In terms of the theory of stationary random processes, Theorem 4 means, in particular, that two stationary and stationarily connected processes with a common correlation function always admit extrapolation from a finite interval with these properties preserved.

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Odessa Civil Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.