ON THE SPECTRAL ANALYSIS OF A ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH PERIODIC COMPLEX-VALUED POTENTIAL

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Abstract

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ON THE SPECTRAL ANALYSIS OF A ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH PERIODIC COMPLEX-VALUED POTENTIAL

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Let T be a closed differential operator in $\mathcal{L}_2(-\infty, \infty)$, generated by the operation

$$-\frac{d^2}{dx^2} + q(x) \tag{1}$$

with a periodic (of period 1) continuous complex-valued function $q(x)$. Denote its domain by $\mathfrak{D}_T$. The aim of the present note is to study the geometric structure of the operator T. We establish that the spectrum of the operator T is separable in the sense of the definition of Yu. I. Lyubich and V. I. Macaev ([1]), and construct a family of bounded projectors onto spectral subspaces possessing some properties of a resolution of the identity. Under additional assumptions on the spectrum, the operator T turns out to be unitarily equivalent to the operator of multiplication by a triangular matrix. The work makes essential use of the method proposed by I. M. Gelfand ([2]) for the case of real $q(x)$. A finite-difference analogue of the operator T was studied by P. B. Naiman ([3]).

1°. With the operator T there is naturally associated a family of operators T_t $(-\pi \leq t \leq \pi)$ in $\mathcal{L}_2(0,1)$, generated by the operation (1) and by the "cyclic" boundary conditions $y(1) = e^{it}y(0)$, $y'(1) = e^{it}y'(0)$. For each t the operator T_t is regular, and its spectrum consists of a countable sequence of eigenvalues tending to infinity. Denote by Σ the union of the spectra of the operators T_t for $-\pi \leq t \leq \pi$.

Theorem 1. *The spectrum of the operator T coincides $*$ with Σ. For every $\lambda \notin \Sigma$ the resolvent of the operator T is an integral operator with kernel*

$$K(x+k, s+n; \lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_t(x, s; \lambda) e^{it(k-n)}\, dt$$

$$(0 \leq x, s \leq 1;\ k, n = 0, \pm 1, \pm 2, ...),$$

where $K_t(x, s; \lambda)$ is the kernel of the resolvent of the operator T_t.

The set Σ, as is known, is the union of the sets of roots of the equations $A(\lambda) = \cos t$ for $-\pi \le t \le \pi$, where $A(\lambda)$ is a certain entire function. Therefore Σ consists of a countable system of bounded simple analytic arcs $\Gamma_1, \Gamma_2, \ldots$, whose endpoints are the roots of the equations $A'(\lambda) = 0$, $A(\lambda) = \pm 1$. Each of the arcs Γ_k can be given by an equation of the form

$$\lambda = \omega_k(\cos t), \tag{2}$$

where ω_k is an invertible function. Let Δ be some closed connected part of one of the arcs Γ_k, containing no points at which $A'(\lambda) = 0$, and let

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$*$ This assertion follows from a result obtained recently by F. S. Rofe-Beketov ([4]).

$\delta \subset [-\pi, \pi]$ is its complete preimage by virtue of equation (2). Denote by $\varphi_t(x)$ and $\psi_t(x)$ the eigenfunctions of the operators T_t and T_t^*, corresponding to the eigenvalues $\lambda(t) = \omega_k(\cos t)$ and $\overline{\lambda(t)}$ $(t \in \delta)$. We normalize them so that

$$\int_0^1 |\varphi_t|^2 \, dx = \int_0^1 \varphi_t \overline{\psi_t} \, dx = 1.$$

We extend each function $\varphi_t(x)$ and $\psi_t(x)$ to the whole axis by the equalities

$$\varphi_t(x+m) = e^{imt}\varphi_t(x), \qquad \psi_t(x+m) = e^{imt}\psi_t(x)$$

$$(0 \le x \le 1, \ m = 0, \pm 1, \pm 2, \ldots).$$

Denote by χ_δ the characteristic function of the set δ, and put, for an arbitrary function $f(x) \in \mathcal{L}_2(-\infty, \infty)$,

$$a(t) = \chi_\delta \int_{-\infty}^{\infty} f(x)\overline{\psi_t(x)} \, dx. \tag{3}$$

Finally, introduce in $\mathcal{L}_2(-\infty, \infty)$ the operator $E(\Delta)$

$$E(\Delta) f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(t)\varphi_t(x) \, dt. \tag{4}$$

Theorem 2. *The operator $E(\Delta)$ is bounded, and*

$$E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2).$$

The operator $E^*(\Delta)$, adjoint to $E(\Delta)$, is defined by the equalities

$$a(t) = \chi_\delta \int_{-\infty}^{\infty} f(x)\overline{\varphi_t(x)}\, dx,$$

$$E^*(\Delta)f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} a(t)\psi_t(x)\, dt.$$

If in Theorem 2 we put $\Delta_1 = \Delta_2 = \Delta$, then we obtain $E^2(\Delta) = E(\Delta)$. Thus $E(\Delta)$ turns out to be a bounded projector onto the subspace $\mathcal{L}_\Delta$ of functions of the form (4). Equality (4) defines the operator V_Δ, which maps $\mathcal{L}_\Delta$ into the subspace Q_δ of functions from $\mathcal{L}_2(-\pi, \pi)$ that are equal to zero outside δ. At the same time the operator V_Δ^{-1} exists and is defined on all of Q_δ.

Theorem 3. *The operator V_Δ is isometric, and for every function $h \in \mathcal{L}_\Delta$ one has*

$$h \in \mathfrak{D}_T, \qquad Th = V_\Delta^{-1}\Lambda_\Delta V_\Delta h,$$

where Λ_Δ is the operator of multiplication by $\lambda(t)$ in Q_δ.

The following theorem establishes the completeness of the system of projectors $E(\Delta)$.

Theorem 4. *Let $E(\Delta)f = 0$ for some function $f \in \mathcal{L}_2(-\infty, \infty)$ for all Δ. Then $f = 0$.*

Next denote by $\mathfrak{M}$ the linear span of all possible functions of the form (4). From Theorems 2 and 4 it is not difficult to obtain the following proposition.

Theorem 5. *The linear manifold $\mathfrak{M}$ is dense in $\mathcal{L}_2(-\infty, \infty)$.*

Operation (1) induces on $\mathfrak{M}$ a certain operator S. Since $S \subset T$, S admits a closure, and moreover $\overline{S} \subseteq T$. Furthermore, the following holds:

Theorem 6. *The operator T coincides with the closure of the operator S.*

Let P be a bounded projector, not necessarily orthogonal. We shall say that it reduces the operator T if its range is contained in $\mathfrak{D}_T$ and is invariant with respect to T, together with the range of the operator $I - P$. It is obvious that if P reduces the operator T, then P and T commute.

Theorem 7. The projector $E(\Delta)$ reduces the operator T for any Δ.

By the preceding theorems, in particular, it has been established that:
1) $\mathcal{L}_\Delta \subset \mathfrak{D}_T$, $T\mathcal{L}_\Delta \subset \mathcal{L}_\Delta$, and 2) the spectrum of the part of the operator T in $\mathcal{L}_\Delta$ coincides with Δ.

Theorem 8. If the subspace R_Δ has properties 1) and 2), then $R_\Delta \subseteq \mathcal{L}_\Delta$.

2°. Suppose now that $A'(\lambda) \neq 0$ for $\lambda \in \Sigma$. In this case the Γ_k have no common points, and each of them is described by the equation $\lambda = \lambda_k(t)$, in which $\{\lambda_k(t)\}_{k=1}^{\infty}$, for fixed t, is the sequence of eigenvalues of the operator T_t. The corresponding sequence of eigenfunctions can, for each t, be transformed into an orthonormal basis $\{u_{n,t}(x)\}_{n=1}^{\infty}$ in $\mathcal{L}_2(0,1)$. In this basis we shall have

$$T_t u_{n,t}(x) = \sum_{i=1}^{n} \lambda_{in}(t) u_{i,t}(x), \qquad \lambda_{ii}(t) = \lambda_i(t),$$

where $\lambda_{in}(t)$ are continuous functions of t on the interval $[-\pi, \pi]$. Denote by Λ the right triangular matrix-function with elements $\lambda_{ik}(t)$ for $i \leq k$. We next introduce into consideration the Hilbert space $\overrightarrow{\mathcal{L}_2}(-\pi, \pi)$ of vector-functions $\alpha = \{a_k(t)\}_{k=1}^{\infty}$, for which

$$\sum_{k=1}^{\infty} \int_{-\pi}^{\pi} |a_k(t)|^2 \, dt < \infty,$$

with scalar product

$$(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} a_k(t) \overline{b_k(t)} \, dt.$$

Define on finite vectors from $\overrightarrow{\mathcal{L}_2}(-\pi, \pi)$ the operator Λ_0 of multiplication by the matrix Λ. It can be shown that Λ_0 admits a closure. The following theorem shows that Λ_0 is a triangular model of the operator T in $\overrightarrow{\mathcal{L}_2}(-\pi, \pi)$.

Theorem 9. The operator T is unitarily equivalent to the closure of the operator Λ_0.

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Note: Figure translations are in progress. See original paper for figures.