I. B. Levinson, P. Sh. Fridberg

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Abstract

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I. B. Levinson, P. Sh. Fridberg

Electromagnetic Coupling of Two Cavities Through a Narrow Slot

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In the well-known works ([1,2]), in solving the integro-differential equation for the field on the slot of two coupled cavities (finite or infinite), the slot is assumed to be exponentially narrow ($\ln \lambda/d \gg 1$, $\ln l/d \gg 1$; l, d are the length and width of the slot, respectively; λ is the wavelength in free space). In the present work the slot is assumed to be simply narrow ($\lambda/d \gg 1$, $l/d \gg 1$), i.e., in the expansion of the kernel of the integral equation, not only terms of order $\ln d$ are retained, but also terms of order 1. A variational-iterative method is presented for obtaining the scattering matrix of two cavities coupled through a narrow slot. The result obtained differs in principle from the expansion used in ([1,2]) in powers of $(\ln \lambda/d)^{-1}$ or $(\ln l/d)^{-1}$.

1. The vector integral equation ([3]) for the electric field on a narrow slot has the form

$$\int_{-d/2}^{d/2} du' \int_0^l dv'\, N_{uu}(uv, u'v')\, E_u(u'v') = H_v^0(uv), \tag{1}$$

where u, v are rectangular coordinates on the slot; u is across the slot from $-d/2$ to $+d/2$; v is along the slot from 0 to l. For a narrow slot E_u may be written in (1) in the form $E_u(u'v') = U(v')\Phi(u')$, where $U(v')$ is the desired voltage on the slot, $\Phi(u') = \pi^{-1}[(d/2)^2 - u'^2]^{-1/2}$ is the known electrostatic function. The use of the indicated electrostatic function is permissible if the slot is located far (at distances much greater than d) from bends of the surface.

Putting $u = 0$ in (1) (the axial line of the slot) and integrating over u', we obtain a one-dimensional integral equation for the voltage on a narrow slot

$$\int_0^l dv'\, N(v, v')\, U(v') = H_v^0(v'). \tag{2}$$

The kernel $N(v, v')$ depends on d and consists of the sum of two averaged admittances corresponding to the coupled cavities ([3])

$$\eta(v, v') = \int du'\, \eta_{uu}(0v, u'v')\, \Phi(u'). \tag{3}$$

Let us separate the dependence of $\eta(v, v')$ on d as $d \to 0$. In doing so we shall neglect terms that tend to zero together with d. As $d \to 0$, the quantity $\eta(v, v')$ tends to infinity as $\ln d$. This is ultimately connected with the singularity of the Green' s functions entering into η. Taking this remark into account, we shall write the input admittance of a cavity, referred to the narrow slot, in the form

$$\eta(v, v') = a^{-1} a(v, v') + b(v, v'), \tag{4}$$

where $\alpha = (2 \ln k_0 d/4)^{-1}$ is a parameter first introduced into the theory of slot antennas by Ya. N. Fel' d, while $a(v, v')$ and $b(v, v')$ do not depend on d, $k_0 = 2\pi/\lambda$. Since the singular part of the Green' s function does not depend on the geometry of the system, irrespective of the geometry of the slot in the volume we have

$$a(v, v') = -\frac{1}{2\pi i} \frac{\eta_0}{k_0} \left[k_0^2 + \frac{\partial^2}{\partial v^2} \right] \delta(v - v'), \tag{5}$$

η_0 is the wave admittance of free space. Explicit expressions for $b(v, v')$ for several special cases will be given below.

Let us note that the quantity of dimension length by which d is divided in the argument of the logarithm is not determined uniquely. Replacing λ by another quantity of the same order, for example l, simply leads to a redefinition of the term $b(v, v')$. The kernel of integral equation (2) takes the form

$$N(v, v') = \alpha^{-1} A(v, v') + B(v, v'), \tag{6}$$

where

$$A(v, v') = 2a(v, v'), \qquad B(v, v') = b_1(v, v') + b_2(v, v'). \tag{7}$$

The terms $b_1(v, v')$ and $b_2(v, v')$ depend on the frequency and the geometry of the coupled volumes.

The presence in (6) of a logarithmic factor makes it possible to consider the approximation of exponentially narrow slots $|\alpha| \ll 1$. In this case one may neglect $B(v, v')$, and the integral equation reduces to Ya. N. Fel' d' s differential equation

$$\left(\frac{\partial^2}{\partial v^2} + k_0^2 \right) U(v) = \pi \alpha (-i\omega\mu_0) H_v^0(v) \tag{8}$$

with the boundary conditions $U(0) = U(l) = 0$, ω being the angular frequency, and μ_0 the magnetic permeability of free space.

2. Let us calculate the nonlogarithmic term $b(v, v')$ for the example of a transverse slot in the broad wall of an infinite multimode waveguide. Using the known [4] tensor Green' s function for an infinite waveguide and differentiating formally, it is easy to find the following expression for the transverse-transverse, relative to the slot, component η_{uu} of the admittance tensor η:

$$\eta(xyz, x'y'z') = \eta_0 \int \frac{dk}{2\pi i} e^{ikz} e^{-ikz'} \sum_{m,n=0}^{\infty} \frac{\varepsilon_m \varepsilon_n}{abk_0} \frac{k_0^2 + \beta_n^2}{\alpha_m^2 + \beta_n^2 + k^2 - k_0^2} \times$$

$$\times \sin \alpha_m x \, \sin \alpha_m x' \cos \beta_n y \, \cos \beta_n y', \tag{9}$$

where xyz are rectangular coordinates: x along a, y along b, z along the waveguide axis; a, b are the dimensions of the cross section of the rectangular waveguide; $\alpha_m = m\pi/a$; $\beta_n = n\pi/b$; $\chi_{mn}^2 = \alpha_m^2 + \beta_n^2$; $\chi_{mnk}^2 = \chi_{mn}^2 + k^2$; $k_{mn} = \sqrt{k_0^2 - \chi_{mn}^2}$ and $q_{mn} = \sqrt{\chi_{mn}^2 - k_0^2}$ are the dispersion laws for propagating and nonpropagating modes, respectively; $\varepsilon_m = 1,\ m = 0$; $\varepsilon_m = 2,\ m = 1, 2, \ldots$. The integration over k is carried out with the poles bypassed in the upper half of the negative real semiaxis and in the lower half of the positive real semiaxis. To calculate (4), we put $z = 0$ in (9) (the axial line of the slot) and average over z' with the aid of $\Phi(z')$. Putting also $y' = 0$, we find

$$\eta(x, x', y) = \eta_0 \int \frac{dk}{2\pi i} \sum_{m,n=0}^{\infty} \frac{\varepsilon_m \varepsilon_n}{abk_0} \frac{k_0^2 + \beta_n^2}{\alpha_m^2 + \beta_n^2 + k^2 - k_0^2} J_0\left(\tfrac{1}{2}kd\right) \times$$

$$\times \sin \alpha_m x \, \sin \alpha_m x' \cos \beta_n y. \tag{10}$$

The integral-series obtained as a result of such a formal procedure is divergent. In order to determine the correct way of summing it, one must examine more carefully the meaning of the limiting

transitions. The twofold formal differentiation of the Green' s function leading to (9) is, strictly speaking, illegal, since the integral series (9) is divergent. To justify this procedure, one may introduce under the sign of the integral-sum a convergence factor
$e^{-\Delta \chi_{mnk}} \sim e^{-\Delta |k|} e^{-\Delta \alpha_m} e^{-\Delta \beta_n}$. This is equivalent to considering an auxiliary Green's function which is the solution of an equation with an auxiliary δ-function smeared over a region with linear dimensions of order Δ. Such a Green's function may be used if all distances under consideration are much larger than Δ. In our problem such a distance is the distance y from the slot to the point where the field is calculated, which must be matched as $y \to 0$. Therefore the limiting passage must be carried out by first letting $\Delta \to 0$, and only then $y \to 0$.

The limiting passage $y \to 0$ must be carried out in the sense $y \to +0$, i.e., taking $y > 0$, since for $y < 0$ (in the exterior region of the waveguide) the fields calculated with the aid of the Green' s function vanish identically. It is not difficult to verify that all this is, in effect, a physical justification of summing the series in the Abel-Poisson sense [5]. Carrying out the summation in (10) by the indicated method, we obtain the desired expression $b(v, v')$ for an infinite multimode waveguide with a transverse slot in the broad wall. The real part $b(x, x')$, characterizing the active part of the admittance, is associated with propagating modes carrying energy to infinity,

$$\operatorname{Re} b(x, x') = \frac{\eta_0}{abk_0} \sum_{m,n=0}^{\chi_{mn}<k_0} \varepsilon_n \frac{k_0^2 - \alpha_m^2}{k_{mn}} \sin\alpha_m x \sin\alpha_m x'. \tag{11}$$

The imaginary part $b(x, x')$, characterizing the reactive part of the admittance, has the form

$$\begin{aligned}\operatorname{Im} b(x, x') = {} & \frac{\eta_0}{\pi k_0} \ln\frac{\lambda}{2b} \left(k_0^2 + \frac{\partial^2}{\partial x^2}\right) \delta(x - x') + \\ & + \frac{\eta_0}{abk_0} \sum_{\alpha_m>k_0}^{\infty} q_{m0} \sin\alpha_m x \sin\alpha_m x' + \frac{2\eta_0}{abk_0} \sum_{m,n=1}^{\chi_{mn}<k_0} \frac{k_0^2 - \alpha_m^2}{\beta_n} \sin\alpha_m x \sin\alpha_m x' + \\ & + \frac{2\eta_0}{abk_0} \sum_{\substack{m,n=1 \\ (\chi_{mn}>k_0)}}^{\infty} (\alpha_m^2 - k_0^2) \left(\frac{1}{q_{mn}} - \frac{1}{\beta_n}\right) \sin\alpha_m x \sin\alpha_m x'.\end{aligned} \tag{12}$$

The presence in this expression of terms of the sum which formally correspond to propagating modes does not mean that these modes contribute to the reactive part of the admittance. Their appearance is connected with transformations made when extracting $\ln d$. Using (10), one can verify that, in the case under consideration, when the slot has no extent in the direction of wave propagation, the propagating modes do not contribute to the reactive part of the admittance. In the case of longitudinal or broad slots, which have an extent in the direction of wave propagation, the propagating modes do make such a contribution.

When calculating the input admittance of a semi-infinite multimode waveguide, it is convenient to abandon the construction of a solenoidal tensor Green' s function, replacing the solenoidality condition $\operatorname{div} G = 0$ by the additional condition $\mathbf{n} \cdot G = 0$ on S. Such a Green' s function is constructed by the method of images, and after taking the double curl its potential part drops out. Differentiating the Green' s function of the semi-infinite waveguide, formally obtained in the manner indicated above, for the component η_{uu} of the admittance tensor η—transverse-transverse relative to the slot in the end face of the waveguide—we obtain

$$\eta(xy, x'y') = \eta_0 \sum_{m,n=0}^{\infty} \frac{\varepsilon_m \varepsilon_n}{abk_0} \frac{k_0^2 - \alpha_m^2}{k_{mn}} \sin\alpha_m x \sin\alpha_m x' \cos\beta_n y \cos\beta_n y'. \quad (13)$$

Set in (13) $y = b/2$, $y' = b/2 + u'$ and split it into two sums corresponding to propagating and nonpropagating modes. Average over u' with the aid of $\Phi(u')$ and pass to the limit $d \to 0$, discarding terms that vanish together with d; we again verify that $\eta(x, x')$ of a semi-infinite waveguide indeed has the form (4). The expressions for $\operatorname{Re} b(x, x')$ and $\operatorname{Im} b(x, x')$ of a semi-infinite waveguide with a slot symmetric with respect to the line $y = b/2$ are obtained from the corresponding expressions for an infinite waveguide by replacing b by $b/2$.

In an analogous way one can obtain expressions for the input admittance η_{uu}, referred to a narrow slot, for resonators and waveguides of other cross-sectional shapes.

3. The coupling of two volumes is completely characterized by the scattering matrix; to determine its elements there is no need at all to solve the integral equation (2). Using the integral equation, by means of the standard procedure one can obtain a variational expression for the elements of the scattering matrix S. In writing the variational principle for the matrix element S_{pq}^{ij}, which gives the coupling of a wave of type i, incident in channel p, with a wave of type j arising in channel q, two trial voltages at the slot, U_p^i and $\tilde{U}_q^j$, naturally enter; these are respectively the voltage at the slot when a wave of type i is incident in channel p, and the voltage at the slot when a wave of type j is incident in channel q. These trial fields may be taken from the solution of the integral equation (2) for $B = 0$. But in this case, as noted above, it becomes the differential equation of J. H. Feld for an exponentially narrow slot. The solution of this equation is known in closed form.

The use, in the variational principle for the scattering matrix, of the voltage on an exponentially narrow slot as the trial voltage should give good results for the following two reasons: first, the variational expression is stationary with respect to small variations of the voltage at the slot; second, the voltage on an exponentially narrow slot satisfies all boundary conditions and is a solution of the integral equation (2) with $B = 0$.

The theory presented is valid for both nonresonant and resonant slots. For a resonant slot, equation (8) gives a sinusoidal field distribution with infinite amplitude; however, as is known, in the variational principle the amplitudes cancel, and equation (8) gives the field distribution for a narrow slot correctly.

In an analogous way, the theory of exponentially thin vibrators ([6,7]) can be generalized to the case of thin vibrators.

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Note: Figure translations are in progress. See original paper for figures.