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Abstract

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MATHEMATICS

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DESCRIPTION OF ANNIHILATOR COMMUTATIVE BANACH ALGEBRAS

(Presented by Academician P. S. Novikov, 14 II 1964)

In the present paper we consider a special case of a very general problem: to describe, up to isomorphism, all commutative Banach algebras (c.b.a.) \mathfrak{A} with prescribed radical \mathfrak{R} and quotient algebra $A = \mathfrak{A}/\mathfrak{R}$.

We shall call \mathfrak{A} **algebraically decomposable** if there exists a subalgebra B in \mathfrak{A} such that $\mathfrak{A} = B \oplus \mathfrak{R}$, and, following ⁽¹⁾, **strongly decomposable** if there exists such a closed subalgebra. In what follows, by an annihilator c.b.a. we shall mean a c.b.a. \mathfrak{A} with one-dimensional radical \mathfrak{R} such that, for any $x \in \mathfrak{A}$, $r \in \mathfrak{R}$, $xr = 0$.

Let \mathfrak{B} be an arbitrary c.b.a. with one-dimensional radical, and let \mathfrak{B}' be the c.b.a. obtained from \mathfrak{B} by adjoining an identity (if \mathfrak{B} itself has an identity, $\mathfrak{B}' = \mathfrak{B}$). It is shown in ⁽²⁾ that in \mathfrak{B}' there exists a maximal ideal which is an annihilator c.b.a.

Thus the case of a c.b.a. with one-dimensional radical is reduced to the case of annihilator c.b.a.'s. Below we give a complete description of algebraically decomposable annihilator c.b.a.'s, proceeding from the properties of the quotient algebra $A = \mathfrak{A}/\mathfrak{R}$.

Let A be a semisimple c.b.a. Introduce the following notation: $S(A)$ is the unit ball of the Banach space A ; T is the convex hull of the set $S(A) \cdot S(A) = \{xy : x, y \in S(A)\}$; $L(T)$ is the linear span of T ; finally, $S = S(A) \cap L(T)$. Thus in $L(T)$, along with the given norm (which will hereafter be called the S -norm), another norm is introduced, specified by the convex centrally symmetric absorbing set T (we shall call it the T -norm). The T -norm is, obviously, no weaker than the S -norm.

Lemma 1. *In order that a linear functional f on $L(T)$ be continuous in the T -norm, it is necessary and sufficient that*

$$|f(xy)| \leq C_1 \|x\| \|y\|$$

for some constant C_1 and all $x, y \in A$.

By $L^*(S)$ and $L^*(T)$ we denote the Banach spaces conjugate to $L(T)$, taken respectively with the S -norm and the T -norm.

Let $f \in L^*(T)$, and let F be its (arbitrary) linear extension to all of A . Introduce in A a new norm $\|\cdot\|_F$, setting, for $x \in A$,

$$\|x\|_F = \max\{\|x\|; |F(x)|\}.$$

With the aid of Lemma 1 one proves

Lemma 2. *The norm $\|\cdot\|_F$ satisfies the multiplicative condition (i.e.*

$$\|xy\|_F \leq C_2 \|x\|_F \|y\|_F$$

for a constant C_2 and all $x, y \in A$).

Consequently, completing A in the norm $\|\cdot\|_F$, we obtain a certain c.b.a.

The following holds.

Theorem 1. *Let the c.b.a. \mathfrak{A} be the completion of A in the norm $\|\cdot\|_F$. Then \mathfrak{A} is semisimple if and only if F is continuous on A , and in this case \mathfrak{A} is isomorphic to A . In the opposite case \mathfrak{A} is an algebraically decomposable annihilator c.b.a. with quotient algebra $\mathfrak{A}/\mathfrak{R}$ isomorphic to A .*

In order to obtain, by means of an arbitrary F , an annihilator c.B.a., we agree, in the case when F is continuous on A , to adjoin directly to A an annihilating one-dimensional radical \mathfrak{R} .

Thus, for any $f \in L^*(T)$, taking its extension F to all of A , we obtain, by the method described, an algebraically decomposable annihilator c.B.a. \mathfrak{A} such that $\mathfrak{A}/\mathfrak{R}$ is isomorphic to A . We shall call it an f -inflation of the algebra A and denote it by $\mathfrak{A}(A, f)$. The correctness of this definition is shown by Lemma 3.

Lemma 3. Let $f \in L^*(T)$; let F_1 and F_2 be two of its extensions to all of A ; and let \mathfrak{A}' and \mathfrak{A}'' be two f -inflations obtained from A with the aid of the functionals F_1 and F_2 . Then \mathfrak{A}' and \mathfrak{A}'' are isomorphic.

We proceed directly to the classification of algebraically decomposable annihilator c.B.a.'s by means of f -inflations. The first step is Lemma 4.

Lemma 4. Let \mathfrak{A} be an arbitrary algebraically decomposable annihilator c.B.a. such that $\mathfrak{A}/\mathfrak{R}$ is isomorphic to A . Then \mathfrak{A} is isomorphic to an f -inflation $\mathfrak{A}_0 = \mathfrak{A}_0(A, f)$ for some $f \in L^*(T)$.

Introduce in the set $L^*(T)$ two equivalence relations:

1°. \sim : $f \sim g$ ($f, g \in L^*(T)$), if, for some constants C_3 and C_4 , $|f(x)| \leq C_3 \max\{\|x\|, |g(x)|\}$ and $|g(x)| \leq C_4 \max\{\|x\|, |f(x)|\}$ for all $x \in L(T)$.

2°. \simeq : $f \simeq g$, if there exists an automorphism ω of the c.B.a. A such that for all $x \in L(T)$ $f(\omega x) = g(x)$.

Denote by \cong the composition of the equivalence relations \sim and \simeq (for the general definition see (3)). The basis of everything that follows is

Lemma 5. Let $\mathfrak{A}_f = \mathfrak{A}_f(A, f)$ and $\mathfrak{A}_g = \mathfrak{A}_g(A, g)$. Then \mathfrak{A}_f and \mathfrak{A}_g are isomorphic if and only if $f \cong g$.

Lemma 5, together with Lemma 4, makes it possible to prove the theorem describing the required class of c.B.a.'s.

Theorem 2. Let A be a semisimple c.B.a. There exists a one-to-one correspondence between the classes of isomorphic algebraically decomposable annihilator c.B.a.'s \mathfrak{A} with $\mathfrak{A}/\mathfrak{R}$, isomorphic to A , on the one hand, and the classes of \cong -equivalent linear functionals from $L^*(T)$, on the other.

The following theorem gives a simple geometric criterion for strong decomposability of annihilator c.B.a.'s.

Theorem 3. Let A be a semisimple c.B.a.; let T and S be the sets defined above. In order that every algebraically decomposable annihilator c.B.a. \mathfrak{A} with $\mathfrak{A}/\mathfrak{R}$, isomorphic to A , be also strongly decomposable, it is necessary and sufficient that, for some constant C_5 , the inclusion $S \subseteq C_5 T$ hold.

Algebras for which such a constant exists will be called algebras possessing the S -property.

As is known, for semisimple c.B.a.'s Gelfand's theorem⁽⁴⁾ holds: algebraically isomorphic semisimple c.B.a.'s are also topologically isomorphic. Theorem 3 makes it possible to describe, among the algebraically decomposable annihilator c.B.a.'s, the class of those possessing an analogous property. Namely, the following holds.

Corollary. Let \mathfrak{A} be an algebraically decomposable annihilator c.B.a. In order that every c.B.a. algebraically isomorphic to \mathfrak{A} be also topologically isomorphic to \mathfrak{A} , it is necessary and sufficient that the factor algebra $A = \mathfrak{A}/\mathfrak{R}$ possess the S -property.

Remark. Trivial examples of c.B.a.'s possessing the S -property are all c.B.a.'s with identity, as well as all closed ideals of the algebras $C(\Omega)$ of all continuous functions on a bicom pactum. A less trivial example is the algebra l_1 of absolutely convergent series with coordinatewise multiplication. On the other hand, the algebras l_p for $1 < p < \infty$, as well as the maximal ideals of the algebras D_n of n -times differentiable functions of one or several variables, certainly do not possess the S -property.

In conclusion we formulate a criterion for strong decomposability of a c.b.a. with an arbitrary one-dimensional radical.

Theorem 4. Let A be a semisimple c.b.a., and let A' be the c.b.a. obtained from A by adjoining an identity (if A itself has an identity, $A' = A$). In order that every algebraically decomposable c.b.a. \mathfrak{A} with one-dimensional radical \mathfrak{R} such that $\mathfrak{A}/\mathfrak{R}$ is isomorphic to A , be also strongly decomposable, it is necessary and sufficient that all maximal ideals of the algebra A' have the S -property.

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