ON REALIZABLE AND COMPLETABLE LOGICO-ARITHMETICAL FORMULAS

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Abstract

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In the article ([1]) S. C. Kleene undertook the development of the principles of the constructive understanding *(realization)* of arithmetical judgments. The concepts introduced by Kleene play an important role in constructive semantics, but in the present note we shall be interested in them from a purely mathematical point of view, outside their semantic purpose, and shall confine ourselves only to revealing a certain curious feature of the relation

$$\text{the object } X \text{ is a realization of the logico-arithmetical formula } F. \quad (1)$$

As for the apparatus that we shall need for the constructive understanding of the assertions occurring below, it can be constructed independently of Kleene' s theory (for example, on the basis of the work of N. A. Shanin ([2])). The preceding phrase, of course, does not mean that all assertions made in the note will be interpreted by us only constructively. On the contrary, we shall be interested to a considerable extent in the discrepancies arising when the truth of judgments is assessed from the standpoint of constructive and of classical logic.

In the literature there are several expositions of Kleene' s theory which differ in technical details but are essentially equivalent. We shall dwell—with minor changes—on the one given in the work of N. A. Shanin ([3]). We shall restore Kleene' s original term "realization," and shall use Shanin' s term "completion" for another purpose. In order to make the reading of this note independent of the work ([3]), we shall give the definition of the relation (1) in full.

Fix some general recursive function (g.r.f.) $\mathfrak{R}$ of two arguments, such that $\mathfrak{R}(m,n) = 0$ if and only if $m = n$. Fix some Gödel numbering of partial recursive functions (p.r.f.) and of pairs of natural numbers. We shall agree to denote the value of a constant term T by the symbol $\zeta(T)$.

We now pass to the definition of the relation (1). The definition will be given by induction on the construction of the formula.

At first we shall consider only constant formulas. Every constant formula, as is known, is uniquely representable in one and only one of the following seven forms: a) $(T = S)$, b) $(P\&Q)$, c) $(P \vee Q)$, d) $(P \supset Q)$, e) $\neg P$, f) $\forall x R(x)$, g)

$\exists x R(x)$ (here T and S are constant terms, P and Q are constant formulas, x is a variable, and $R(x)$ is a formula containing no variables distinct from x).

Case a). Only a g.r.f. $\mathfrak{R}$ may occur as a realization of the formula. The g.r.f. $\mathfrak{R}$ is considered a realization of the formula $(T = S)$ if and only if $\mathfrak{R}(\zeta(T), \zeta(S)) = 0$. The record of the realization $\mathfrak{R}$ of the formula $(T = S)$ is considered to be the Gödel number (g.n.) $\mathfrak{R}$.

Case b). Only pairs of natural numbers may occur as realizations of the formula. A pair (a, b) is considered a realization of the formula $(P \& Q)$ if and only if a is the record of some realization of the formula P, and b is the record of some realization of the formu-

of Q. The record of the realization (a, b) of the formula $(P \& Q)$ is regarded as the g.n. of the pair (a, b).

Case c). Only pairs of natural numbers of the form $(1, b)$ or $(2, b)$ may occur as realizations. The pair $(1, b)$ is regarded as a realization of the formula $(P \vee Q)$ if and only if b is the record of some realization of P. The pair $(2, b)$ is regarded as a realization of the formula $(P \vee Q)$ if and only if b is the record of some realization of Q. The record of the realization (a, b) of the formula $(P \vee Q)$ is regarded as the g.n. of the pair (a, b).

Case d). Only partial recursive functions of one argument may occur as realizations of the formula. A partial recursive function φ is regarded as a realization of the formula $(P \supset Q)$ if and only if φ is applicable to every number that is the record of some realization of the formula P, and transforms every such number into the record of some realization of the formula Q. The record of the realization φ of the formula $(P \supset Q)$ is regarded as the g.n.f.

Case e). Only partial recursive functions of one argument may occur as realizations. A partial recursive function φ is a realization of the formula $\neg P$ if and only if it is a realization of the formula $(P \supset (0 = 1))$. The record of the realization φ of the formula $\neg P$ is regarded as the g.n.f.

Case f). Only general recursive functions of one argument may occur as realizations. A general recursive function φ is regarded as a realization of the formula $\forall x R(x)$ if and only if, for every n, $\varphi(n)$ is the record of some realization of the formula $R(n)$. The record of the realization φ of the formula $\forall x R(x)$ is regarded as the g.n. φ.

Case g). Only pairs of natural numbers may occur as realizations of the formula. The pair (a, b) is regarded as a realization of the formula $\exists x R(x)$ if and only if a is the record of some realization of the formula $R(b)$. The record of the realization (a, b) of the formula $\exists x R(x)$ is regarded as the g.n. of the pair (a, b).

The realizations of a formula F with free variables (by definition) coincide with the realizations of the closure of the formula F.

A formula is called **realizable** if there exists an object that is its realization.

Let us now define the relation "the object X is a **completion** of the logical-arithmetical formula F." In its structure this definition will completely coincide with definition of relation (1). The essential difference will consist only in the fact that the last phrase of each of items a)–g)*) of the new definition will have the form: **the record of the completion** W **of the formula** F **is regarded as the natural number** 0. (Of course, the term "realization" must everywhere be replaced by the term "completion," with the proper grammatical agreement.) In view of the noted complete analogy of the definitions, we shall restrict ourselves to formulating (as examples) items a), b), d), and f).

Case a). Only a general recursive function $\mathfrak{R}$ may occur as a completion of the formula. A general recursive function $\mathfrak{R}$ is regarded as a completion of the formula $(T = S)$ if and only if $\mathfrak{R}(\mathfrak{z}(T), \mathfrak{z}(S)) = 0$. The record of the completion $\mathfrak{R}$ of the formula $(T = S)$ is regarded as the natural number 0.

Case b). Only pairs of natural numbers may occur as completions of the formula. The pair (a, b) is regarded as a completion of the formula $(P\&Q)$ if and only if a is the record of some completion of P, and b is the record of some completion of the formula Q. The record of the completion (a, b) of the formula $(P\&Q)$ is regarded as the natural number 0.

Case d). Only partial recursive functions of one argument may occur as completions of the formula. A partial recursive function φ is regarded as a completion of the formula $(P \supset Q)$ if and only if φ is applicable to every number that is the record of some completion of the formula P, and transforms every such number into the record of some completion of the formula

formula Q. The record of a fulfillment φ of the formula $(P \supset Q)$ is considered to be the natural number 0.

Cases e). Only g.r.f.' s of one argument may occur as fulfillments. A g.r.f. φ is considered to be a fulfillment of the formula $\forall x R(x)$ if and only if, for each n, $\varphi(n)$ is the record of some fulfillment of the formula $R(n)$. The record of a fulfillment φ of the formula $\forall x R(x)$ is considered to be the natural number 0.

The remaining clauses are formulated analogously. Fulfillments of formulas with free variables are defined as fulfillments of the closures of these formulas.

A formula is called **fulfillable** if there exists an object that is its fulfillment.

Thus, the difference between the definitions of a realizable and a fulfillable logical-arithmetical formula consists in the manner of defining the record of a realization and of a fulfillment. Whereas in the record of a realization the "information" about the realization itself is preserved completely, in the record of a fulfillment the "information" about the fulfillment is entirely absent.

There arises the question of comparing the sets of realizable and fulfillable logical-arithmetical formulas. Let us denote by the symbol $\mathfrak{H}$ the following property of a logical-arithmetical formula P: P is fulfillable if and only if it is realizable.

Within the framework of the constructive understanding of judgments, the following can be proved:

Theorem 1. *Every logical-arithmetical formula has property* $\mathfrak{H}$.

Within the framework of classical logic, the following can be proved:

Theorem 2. *There is no algorithm that recognizes logical-arithmetical formulas possessing property* $\mathfrak{H}$.

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CITED LITERATURE

1. S. C. Kleene, *J. Symbolic Logic*, **10**, 109 (1945).
2. N. A. Shanin, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **52**, 226 (1958).
3. N. A. Shanin, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **43** (1955).

Note: Figure translations are in progress. See original paper for figures.