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Abstract

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ESTIMATING THE COMPLEXITY OF LOCAL ALGORITHMS FOR SOME EXTREMAL PROBLEMS ON FINITE SETS

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There exists a class of problems (the problem of constructing the shortest path between vertices of a graph, the problem of constructing minimal d.n.f.' s for functions of the algebra of logic, etc.) whose solution requires selecting, from a finite set of objects, those objects that possess certain extremal properties. Thus, in constructing minimal d.n.f.' s realizing a function f, it is necessary to select, in the reduced d.n.f. $\mathfrak{M}_f$, the conjunctions that do not occur in any minimal d.n.f.; in constructing a shortest path between two given vertices of a graph, it is necessary to select the edges of the graph that occur in at least one shortest path; and so on.

Usually the computation of such extremal properties requires algorithms of high laboriousness. However, for problems of this type the very concept of "laboriousness" of an algorithm has not been precisely formulated, and therefore there are no sufficiently good estimates of this quantity.

In note (^{1}) a definition was given and certain properties were formulated for algorithms that compute information about elements of finite sets. At each step such an algorithm, using previously accumulated information about the neighborhood $S(\mathfrak{A}, \mathfrak{M})$ of an element $\mathfrak{A}$ from a set $\mathfrak{M}$, attempts to compute the value of one property from a previously fixed set of predicates $\{P(\mathfrak{A}, \mathfrak{M})\}$. The algorithms defined in (^{1}) will be called **local**. It is natural to characterize local algorithms by two parameters: the cardinality of the set $\{P(\mathfrak{A}, \mathfrak{M})\}$ and the size of the neighborhood $S(\mathfrak{A}, \mathfrak{M})$. In the present note a definition of these parameters is given, and it is shown that algorithms computing certain extremal properties of conjunctions from reduced d.n.f.' s and of edges of a graph have very large values of the parameters.

I. Index of an algorithm A**.** Let a family $\{\mathfrak{M}\}$ of finite sets be given. To each pair $(\mathfrak{A}, \mathfrak{M})$ we assign a sequence

$$S_1(\mathfrak{A}, \mathfrak{M}), \dots, S_r(\mathfrak{A}, \mathfrak{M}), \dots$$

satisfying the following conditions: 1°. $S_i(\mathfrak{A}, \mathfrak{M})$ is a neighborhood (^{1}) of $\mathfrak{A}$ in $\mathfrak{M}$. 2°.

$$S_1(\mathfrak{A}, \mathfrak{M}) \subseteq S_2(\mathfrak{A}, \mathfrak{M}) \subseteq \cdots \subseteq S_r(\mathfrak{A}, \mathfrak{M}) \subseteq \cdots.$$

3°. For all i there exists a pair $(\mathfrak{A}_i, \mathfrak{M}_i)$, $\mathfrak{A}_i \in \mathfrak{M}_i$, $\mathfrak{M}_i \in \{\mathfrak{M}\}$, such that

$$S_i(\mathfrak{A}_i, \mathfrak{M}_i) \subset S_{i+1}(\mathfrak{A}_i, \mathfrak{M}_i).$$

An algorithm A belonging to the class $K(P_1, \dots, P_k, P_{i_1}, \dots, P_{i_l}, \varphi_1, \dots, \varphi_k)$ will be called an **algorithm of index** r if the domain of definition of the functions φ_i, $i = 1, 2, \dots, k$, is

$$\langle \mathfrak{A}^{\alpha_1 \cdots \alpha_k}, S_r(\mathfrak{A}^{\alpha_1 \cdots \alpha_k}, \mathfrak{M}^*) \rangle, \quad M(\mathfrak{M}^*) \in \{\mathfrak{M}\},$$

and all marks in $S_r(\mathfrak{A}^{\alpha_1 \cdots \alpha_k}, \mathfrak{M}^*)$ are admissible (^{1}).

II. Let $\mathfrak{A}$ be sets and let $\mathfrak{M} \in \{\mathfrak{M}\}$ be a collection of sets. We introduce the sequence

$$S_1(\mathfrak{A}, \mathfrak{M}), \dots, S_r(\mathfrak{A}, \mathfrak{M}), \dots$$

as follows: $S_1(\mathfrak{A}, \mathfrak{M})$ is made up of all elements $\mathfrak{B}$ satisfying one of the conditions: 1°. $\mathfrak{A} \cap \mathfrak{B}$ is nonempty. 2°. Let $\mathfrak{A}_1, \dots, \mathfrak{A}_q$ satisfy 1°. Then

$$\mathfrak{B} \subseteq \bigcup_{i=1}^{q} \mathfrak{A}_i.$$

Suppose the set $S_{r-1}(\mathfrak{A}, \mathfrak{M})$ has been defined. We form $S_r(\mathfrak{A}, \mathfrak{M})$ from all elements $\mathfrak{B}$ satisfying one of the conditions: 1°. There is an element $\mathfrak{A}_i$, $\mathfrak{A}_i \in S_{r-1}(\mathfrak{A}, \mathfrak{M})$, such that $\mathfrak{A}_i \cap \mathfrak{B}$ is nonempty. 2°. Let $\mathfrak{B}_1, \dots, \mathfrak{B}_q$ satisfy 1°. Then

$$\mathfrak{B} \subseteq \bigcup_{i=1}^{q} \mathfrak{B}_i.$$

The set $S_r(\mathfrak{A}, \mathfrak{M})$ will be called the **principal neighborhood of order** l of the element $\mathfrak{A}$ in the set $\mathfrak{M}$.

We shall also consider the sequences

$$S'_1(\mathfrak{A}, \mathfrak{M}), \dots$$

$\cdots$, $S'_r(\mathfrak{A}, \mathfrak{M}), \dots$ such that $S_{i-1}(\mathfrak{A}, \mathfrak{M}) \subseteq S'_i(\mathfrak{A}, \mathfrak{M}) \subseteq S_i(\mathfrak{A}, \mathfrak{M})$ and there exists a pair $(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{M}})$ such that $S_{i-1}(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{M}}) \subset S'_i(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{M}})$.

We shall call an algorithm A an **index algorithm** if the functions φ_i, $i = 1, 2, \dots, k$, are defined on pairs $\langle \mathfrak{A}^{\alpha_1 \cdots \alpha_k}, S'_r(\mathfrak{A}^{\alpha_1 \cdots \alpha_k}, \mathfrak{M}^*) \rangle$, $M(\mathfrak{M}^*) \in \{\mathfrak{M}\}$, and all labels in $S'_r(\mathfrak{A}^{\alpha_1 \cdots \alpha_k}, \mathfrak{M}^*)$ are admissible.

III. Let f be a function of the algebra of logic, $\mathfrak{M}_f$ the set of conjunctions entering into the reduced disjunctive normal form of the function f, and N^f the set of maximal intervals of the function f. The definitions of II can obviously be applied to the pairs $(\mathfrak{A}, \mathfrak{M}_f)$, $\mathfrak{A} \in \mathfrak{M}_f$, and $(N_{\mathfrak{A}}, N^f)$, $N_{\mathfrak{A}} \in N^f$ (2).

Let

$$\Gamma = [\langle a_1, \dots, a_s \rangle,\ \langle (a_{i_1}, a_{i_2}), \dots, (a_{i_m}, a_{i_l}) \rangle]$$

be an undirected graph, where $a_1, \dots, a_s$ are the vertices of the graph and $(a_{i_1}, a_{i_2}), \dots, (a_{i_m}, a_{i_l})$ are the edges of the graph. Applying the definitions of II, we obtain the definitions of the principal neighborhoods (2) of order r of the vertices and edges of a graph and the definition of an index algorithm of order r on the vertices and edges of a graph.

IV. Let the algorithm A belong to the class $K(P_1, \dots, P_k, P_i, \dots, P_{i_l}, \dots, \varphi_1, \dots, \varphi_k)$. The number k will be called the **memory size** of the algorithm A.

V. We pass to the notion of computability of a predicate in the class of local algorithms. Usually, when solving concrete problems, certain restrictions are imposed on the predicates $P_1, \dots, P_k$ and the functions $\varphi_1, \dots, \varphi_k$. We shall assume that certain sets $\{P(\mathfrak{A}, \mathfrak{M})\}$ and $\{\varphi\}$ are given and that all predicates $P_1, \dots, P_k$ and functions $\varphi_1, \dots, \varphi_k$ participating in the definition of local algorithms are chosen respectively from $\{P(\mathfrak{A}, \mathfrak{M})\}$ and $\{\varphi\}$.

Definition. The predicate $P_1(\mathfrak{A}, \mathfrak{M}) \in \{P(\mathfrak{A}, \mathfrak{M})\}$ is called (r, k)**-computable** if there exists an algorithm A, belonging to the class $K(P_1, \dots, P_k, P_1, \varphi_1, \dots, \varphi_k)$, such that: 1°. $P_i \in \{P(\mathfrak{A}, \mathfrak{M})\}$, $i = 1, 2, \dots, k$. 2°. $\varphi_i \in \{\varphi\}$, $i = 1, 2, \dots, k$. 3°. For all $\mathfrak{M} \in \{\mathfrak{M}\}$, in $A(\mathfrak{M})$ the label vectors of all elements have first coordinate different from Δ.

VI. Let P_2 be the set of all functions of the algebra of logic, and $P_2(n)$ the set of functions of the algebra of logic depending on n variables. Denote by $\{\mathfrak{M}_f\}$ the set of reduced disjunctive normal forms of all functions of the algebra of logic and by $\{\mathfrak{M}_f^n\}$ the set of reduced disjunctive normal forms of all functions of the algebra of logic in n variables. To each DNF $\mathfrak{M}_f$ there corresponds one-to-one the set $\mathfrak{M}_f$ of all conjunctions from $\mathfrak{M}_f$. We shall consider the sets $\{\mathfrak{M}_f\}$ and $\{\mathfrak{M}_f^n\}$.

We introduce restrictions on the set $\{P(\mathfrak{A}, \mathfrak{M})\}$. Consider transformations $\{\pi\}$ of the variables $x_i \to x_j^{\sigma}$. These transformations were studied by K. Shannon (3) and G. N. Povarov (4). The transformation π induces a transformation on the set of conjunctions $\pi(x_{i_1}^{\sigma_1} \cdot \dots \cdot x_{i_k}^{\sigma_k}) = \pi(x_{i_1}^{\sigma_1}) \cdot \dots \cdot \pi(x_{i_k}^{\sigma_k})$ and on the set of DNFs $\pi(\mathfrak{A}_1 \vee \dots \vee \mathfrak{A}_l) = \pi(\mathfrak{A}_1) \vee \dots \vee \pi(\mathfrak{A}_l)$. We shall call the predicate $P(\mathfrak{A}, \mathfrak{M})$ **invariant with respect to** $\{\pi\}$ if for every $\pi \in \{\pi\}$ the equality

$$P(\mathfrak{A}, \mathfrak{M}_f) = P(\pi(\mathfrak{A}), \pi(\mathfrak{M}_f)), \qquad \mathfrak{A} \in \mathfrak{M}_f, \quad \mathfrak{M}_f \in \{\mathfrak{M}_f\}$$

holds.

The set of predicates $P(\mathfrak{A}, \mathfrak{M}_f)$ invariant with respect to $\{\pi\}$ will be denoted by $P(\pi)$.

Consider transformations $\{\overline{\pi}\}$ of reduced DNFs: to a set of conjunctions $\{\mathfrak{A}_i\}$ forming a reduced DNF we assign a set of conjunctions $\{\mathfrak{B}_i\}$, $(\{\mathfrak{B}_i\} = \overline{\pi}\{\mathfrak{A}_i\})$, so that the following conditions are satisfied: 1°. To each conjunction $\mathfrak{A}_i$ from

$\{\mathfrak{A}_i\}$ there corresponds one-to-one a conjunction of the same rank $\mathfrak{B}_i$ from $\{\mathfrak{B}_i\}$ $(\mathfrak{B}_i = \bar{\pi}(\mathfrak{A}_i))$. 2°. If $N_{\mathfrak{A}} \subseteq \bigcup_{i=1}^{l} N_{\mathfrak{A}_i}$,

then

$$N_{\bar{\pi}}(\mathfrak{A}) \subseteq \bigcup_{i=1}^{l} N_{\bar{\pi}}(\mathfrak{A}_i).$$

3°. If

$$N_{\mathfrak{A}} \nsubseteq \bigcup_{i=1}^{l} N_{\mathfrak{A}_i},$$

then

$$N_{\bar{\pi}}(\mathfrak{A}) \nsubseteq \bigcup_{i=1}^{l} N_{\bar{\pi}}(\mathfrak{A}_i),$$

$\mathfrak{A}_i \in \{\mathfrak{A}_i\},\ i = 1, 2, \dots, l.$

We shall call a predicate $P(\mathfrak{A}, \mathfrak{M}_f)$ **invariant with respect to** $\{\bar{\pi}\}$ if, for every $\bar{\pi} \in \{\bar{\pi}\}$, the relation

$$P(\mathfrak{A}, \mathfrak{M}_f) = P(\bar{\pi}(\mathfrak{A}), \bar{\pi}(\mathfrak{M}_f))$$

holds. The set of predicates $P(\mathfrak{A}, \mathfrak{M}_f)$ invariant with respect to $\{\bar{\pi}\}$ will be denoted by $P(\bar{\pi})$. The transformations π are a special case of the transformations $\bar{\pi}$.

VII. Consider the predicates: $P_1(\mathfrak{A}, \mathfrak{M}_f)$ $(P_1'(\mathfrak{A}, \mathfrak{M}_f))$– "$\mathfrak{A}$ enters at least one minimal (shortest) d.n.f. of the function f" ; $P_2(\mathfrak{A}, \mathfrak{M}_f)$ $(P_2'(\mathfrak{A}, \mathfrak{M}_f))$– "$\mathfrak{A}$ enters all minimal (shortest) d.n.f.' s of the function f" ; $P_3(\mathfrak{A}, \mathfrak{M}_f)$ $(P_3'(\mathfrak{A}, \mathfrak{M}_f))$– "$\mathfrak{A}$ enters minimal (shortest) d.n.f.' s of the function f, but not all of them" ; $P_4(\mathfrak{A}, \mathfrak{M}_f)$ $(P_4'(\mathfrak{A}, \mathfrak{M}_f))$– "$\mathfrak{A}$ enters at least one irredundant d.n.f. composed of the greatest number of letters (conjunctions)" ; $P_5(\mathfrak{A}, \mathfrak{M}_f)$ $(P_5'(\mathfrak{A}, \mathfrak{M}_f))$– "$\mathfrak{A}$ enters all irredundant d.n.f.' s composed of the greatest number of letters (conjunctions)" ; $P_6(\mathfrak{A}, \mathfrak{M}_f)$ $(P_6'(\mathfrak{A}, \mathfrak{M}_f))$– "$\mathfrak{A}$ enters irredundant d.n.f.' s composed of the greatest number of letters (conjunctions), but not all of them."

Lemma. The predicates $P_i(\mathfrak{A}, \mathfrak{M}_f)$, $P_i'(\mathfrak{A}, \mathfrak{M}_f)$, $i = 1, 2, 3, 4, 5, 6$, are invariant with respect to $\{\pi\}$ and $\{\bar{\pi}\}$.

VIII. Consider the set $\{\mathfrak{M}_f^n\}$ (see VI) and algorithms A such that: 1°. $A \in K(P^1, \dots, P^k, P_\alpha, \varphi_1, \dots, \varphi_k)$. 2°. The index of the algorithm A is equal to

r. 3°. $P_\alpha \in \{P_i(\mathfrak{A}, \mathfrak{M}_f), P_i'(\mathfrak{A}, \mathfrak{M}_f)\}$, $i = 1, 2, 3, 4, 5, 6$ (see VII). $P^1, \dots, P^k$ belong to $P(\pi)$. 4°. $\varphi_i = \varphi_i(\mathfrak{A}^{\alpha_1 \cdots \alpha_k}, S(\mathfrak{A}^{\alpha_1 \cdots \alpha_k}, \mathfrak{M}_f^*))$.

Theorem 1. If $r < \infty$ and $k < \infty$, then the predicate P_α is not (r, k)-computable.

IX. Consider the set $\{\mathfrak{M}_f^n\}$ and algorithms A such that: 1°. $A \in K(P^1, \dots, P^k, P_\alpha, \varphi_1, \dots, \varphi_k)$. 2°. The index of A is equal to r. 3°. $P_\alpha \in \{P_i(\mathfrak{A}, \mathfrak{M}_f), P_i'(\mathfrak{A}, \mathfrak{M}_f)\}$, $i = 1, 2, 3, 4, 5, 6$ (see VII). $P^1, \dots, P^k$ belong to $P(\bar{\pi})$.

Theorem 2. For every $n > n_0$ there exists $\varepsilon > 0$, $\lim_{n\to\infty} \varepsilon(n) = 0$, such that, if

$$rk < \frac{2^n}{B(n)}(1 - \varepsilon),$$

then the predicate P_α is not (r, k)-computable. (Here $B(n)$ is a function growing more slowly than the k-fold logarithm of n for any k.)

In proving Theorem 2, the following theorem, established by Yu. L. Vasil' ev (5,6), is used essentially: in the set $P_2(n)$ there exists a cycle of length

$$\frac{2^n}{B(n)}(1 - \varepsilon), \qquad \varepsilon \to 0 \quad \text{as } n \to \infty.$$

X. Let a bipartite graph $\Gamma(a_i, a_j) = M_1 \cup M_2$ be given, where $M_1 = \{a_1, \dots, a_k\}$, $M_2 = \{(a_{i_1}, a_{i_2}), \dots, (a_l, a_s)\}$, the number of elements in M_2 is not greater than n; a_i, a_j are the poles of the graph. We shall denote by $\Gamma_2(n)$ the set of graphs with two poles and with a number of edges not exceeding n.

Graphs $\Gamma(a_i, a_j) = M_1 \cup M_2$ and $\Gamma'(b_p, b_q) = M_1' \cup M_2'$ will be called **isomorphic** if there exists a one-to-one mapping φ of the set M_1 onto the set M_1' such that: 1°. $\langle b_p, b_q \rangle = \langle \varphi(a_i), \varphi(a_j) \rangle$. 2°. If (a_m, a_n) is an edge of the graph Γ, then $(\varphi(a_m), \varphi(a_n))$ is an edge of the graph Γ'. 3°. If (a_m, a_n) is not an edge of Γ, then $(\varphi(a_m), \varphi(a_n))$ is not an edge of the graph Γ'.

In what follows we shall consider properties of vertices and edges of a graph that take identical values on the corresponding elements of isomorphic graphs. Denote by $P(J)$ the set of predicates $P(a, \Gamma)$, $P((a_m, a_n), \Gamma)$ satisfying the following condition: if $\varphi(\Gamma) = \Gamma'$, then $P(a, \Gamma) = P(\varphi(a), \Gamma')$, $P((\varphi(a_m), \varphi(a_n)), \Gamma') = P((a_m, a_n), \Gamma)$. In defining local algorithms on graphs, we shall consider only predicates from the set $P(J)$.

XI. A set of edges of the graph $\Gamma(a, b)$ forms a path between the poles if it contains a sequence $(a_1, a_i), \dots, (a_j, b)$. A path is called a dead end if it contains no subpaths. A path is called minimal if it consists of the minimal number of edges (among all paths of the graph). The predicates $\widetilde{P}_1(a_j, \Gamma)$

$(\widetilde{P}_1(R,\Gamma))$, "the vertex a_j (edge R) belongs to at least one dead-end path of the graph Γ" ; $\widetilde{P}_2(a_j,\Gamma)$ $(\widetilde{P}_2(R,\Gamma))$, "the vertex a_j (edge R) belongs to at least one minimal path of the graph Γ," are, obviously, invariant under isomorphic mappings of the graph Γ, and therefore are contained in $P(J)$. The predicate $\widetilde{P}_1$ is an analogue of the predicate "the conjunction $\mathfrak{A}$ belongs to at least one dead-end d.n.f. of the function f." In [2] it was proved (in other terms) that the latter predicate is $(2,1)$-computable. It turns out, however, that even computing the property $\widetilde{P}_1$ for vertices and edges of a graph requires algorithms with large r and k.

XII. We shall assume that, before the algorithm begins to operate, on all vertices and edges of the graph the predicate $P_1(\mathfrak{A},\Gamma)$, "$\mathfrak{A}$ is a pole of the graph," has been computed.

$$P_1(\mathfrak{A},\Gamma)(\mathfrak{A}_i,a_j)=1,\quad \text{if } \mathfrak{A} \text{ is a vertex of the graph and } \mathfrak{A}\in\{a_i,a_j\}.$$

$P_1(\mathfrak{A},\Gamma)=0$ on all the remaining vertices of the graph $\Gamma(a_i,a_j)$ and all edges of the graph $\Gamma(a_i,a_j)$.

XIII. Some of the predicates $P\in P(J)$ have meaning simultaneously for vertices and edges of the graph, others only for vertices or only for edges. If a predicate P has meaning only for edges, then we shall assume that $P(a,\Gamma)\equiv 0$. We extend similarly predicates that have meaning only for vertices of the graph.

XIV. Let $\mathfrak{A},\mathfrak{B}$ be elements of the graphs Γ and Γ' (edges or vertices). We shall call the neighborhoods $S(\mathfrak{A}^{\alpha_1\cdots\alpha_k},\Gamma^*)=S$ and $S'=S(\mathfrak{B}^{\alpha_1\cdots\alpha_k},\Gamma'^*)$ isomorphic if there exists a mapping φ that establishes a one-to-one correspondence between the vertices belonging to S and S', the edges belonging to S and S', with

$$\varphi(\mathfrak{A})=\mathfrak{B}\quad\text{and}\quad\varphi(\mathfrak{A}^{\gamma_1\cdots\gamma_k})=\mathfrak{B}^{\gamma_1\cdots\gamma_k}.$$

In defining local algorithms over graphs we shall consider monotone functions φ_i satisfying the following condition: if $S(\mathfrak{A}^{\alpha_1\cdots\alpha_k},\Gamma^*)$ and $S(\mathfrak{B}^{\alpha_1\cdots\alpha_k},\Gamma'^*)$ are isomorphic neighborhoods, then

$$\varphi_i(\mathfrak{A}^{\alpha_1\cdots\alpha_k},S(\mathfrak{A}^{\alpha_1\cdots\alpha_k},\Gamma^*))=\varphi_i(\mathfrak{B}^{\alpha_1\cdots\alpha_k},S(\mathfrak{B}^{\alpha_1\cdots\alpha_k},\Gamma'^*)).$$

XV. Consider the set $\Gamma_2(n)$ and an algorithm A whose predicates satisfy the restrictions of X (are contained in $P(J)$) and whose functions φ_i satisfy the restriction of XIV.

Theorem 3. There exists a constant β such that, if

$$r<\beta n(1-\varepsilon),\quad \varepsilon\to 0 \text{ as } n\to\infty,\quad k<\infty,$$

then the predicate $\widetilde{P}_1(a,\Gamma)$, $(\widetilde{P}_1(R,\Gamma))$, "the vertex a (edge R) belongs to at least one dead-end path between the poles of the graph Γ," is not (r,k)-computable.

Theorem 4. There exists a constant β', $\beta' > \beta$, such that, if

$$r < \beta' n(1-\varepsilon), \quad \varepsilon \to 0 \text{ as } n \to \infty, \quad k < \infty,$$

then the predicate $\widetilde{P}_2(a, \Gamma)$ $(\widetilde{P}_2(R, \Gamma))$, "the vertex a (edge R) belongs to at least one minimal path between the poles of the graph Γ," is not (r, k)-computable.

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Note: Figure translations are in progress. See original paper for figures.

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