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Abstract

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BOUNDARY-VALUE PROBLEMS FOR GENERAL SINGULAR EQUATIONS IN A BOUNDED DOMAIN

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1. In a bounded domain $G \subset R^n$ with a sufficiently smooth boundary Γ, an equation of the form

$$K\varphi \equiv K_\alpha\varphi + T\varphi = \int_G K_\alpha(x, x-y)\varphi(y)\,dy + \int_G T(x,y)\varphi(y)\,dy = F(x), \quad (1)$$

$x \in G$, is considered. Here $K_\alpha(x,z)$, $T(x,z)$ are, generally speaking, generalized functions in z, depending smoothly on x, and the integrals in (1) are understood in the sense of the theory of generalized functions.

It is assumed that: a) the symbol $\widetilde{K}_\alpha(x,\xi)$ (the Fourier transform of $K_\alpha(x,z)$ with respect to z, $FK_\alpha(x,z) = \widetilde{K}_\alpha(x,\xi)$, is a homogeneous function of ξ of order α, where α may be any number); b) an analogue of the ellipticity condition is satisfied: $\widetilde{K}_\alpha(x,\xi) \neq 0$ for all $\xi \neq 0$, $x \in G \cup \Gamma$. For simplicity we assume that $\widetilde{K}_\alpha(x,\xi)$ is an infinitely differentiable function of x and ξ ($\xi \neq 0$, $x \in G \cup \Gamma$). Obviously,

$$\widetilde{K}_\alpha(x,\xi) = \sum_{|\gamma|=0}^{\infty} a_\gamma(x)\widetilde{Z}^{(\gamma)}(\xi),$$

where

$$\widetilde{Z}^{(\gamma)}(\xi) = \widetilde{Y}^{(\gamma)}(\xi)|\xi|^\alpha,$$

and $\widetilde{Y}^{(\gamma)}(\xi)$, for example, are spherical functions (cf. ([1,2])). Consequently,

$$K_\alpha\varphi = \sum a_\gamma(x)Z^{(\gamma)} * \varphi_+ \quad (x \in G),$$

where $\varphi_+(x) = \varphi(x)$ for $x \in G \cup \Gamma$, $\varphi_+(x) = 0$ for $x \notin G \cup \Gamma$, and

$$Z^{(\gamma)} * \varphi_+ = F^{-1}\big(\widetilde{Z}^{(\gamma)}(\xi)\widetilde{\varphi}_+(\xi)\big).$$

In the case $\alpha = 0$, (1) is a singular integral equation in the bounded domain G. In particular, (1) may be an elliptic differential equation. However, equation (1) also includes elliptic integro-differential equations and the case when $\widetilde{K}_\alpha(x, \xi)$, for example, is a rational function of ξ.

2. Let us factorize the kernel $\widetilde{K}_\alpha(x_0, \xi)$ at a point $x_0 \in \Gamma$. For this, choose at this point such a coordinate system (x', x_n) that $x_n > 0$ is the direction of the inner normal to Γ, and $x_n = 0$ is the equation of the tangent plane to Γ at the point x_0, and represent $\widetilde{K}_\alpha(x_0, \xi', \xi_n)$ in the form

$$\widetilde{K}_\alpha(x_0, \xi', \xi_n) = \frac{K^+_\varkappa(x_0, \xi', \xi_n)}{K^-_{\varkappa-\alpha}(x_0, \xi', \xi_n)} \quad (\xi = (\xi', \xi_n)), \tag{2}$$

where $K^+_\varkappa$, $K^-_{\varkappa-\alpha}$ are analytic functions of ξ_n in the half-planes $\operatorname{Im}\xi_n > 0$, $\operatorname{Im}\xi_n < 0$, respectively, with $K^+_\varkappa \neq 0$ for $\operatorname{Im}\xi_n \geq 0$, $K^-_{\varkappa-\alpha} \neq 0$ for $\operatorname{Im}\xi_n \leq 0$, $\xi \neq 0$. $K^+_\varkappa$ and $K^-_{\varkappa-\alpha}$ are, up to a bounded nonzero factor, homogeneous functions of $\xi = (\xi', \xi_n)$ of orders $\varkappa$ and $\varkappa - \alpha$. The number $\varkappa = \varkappa(x_0)$ is called the **index** of the kernel K_α at the point $x_0 \in \Gamma$, and

$$\varkappa = \frac{\alpha}{2} + \Delta_n \arg \widetilde{K}_\alpha(x_0, \xi', \xi_n),$$

where $\Delta_n \arg \widetilde{K}_\alpha(x_0, \xi', \xi_n)$ is the increment of the argument of $K_\alpha(x_0, \xi', \xi_n)$ as ξ_n varies from $+\infty$ to $-\infty$ (ξ' fixed) (see [3–5]). Continue the function $\varkappa(x)$, defined for $x \in \Gamma$, continuously into G, and take a sufficiently fine finite covering $\{U_j\}$ of the domain $G \cup \Gamma$ so that the oscillation of $\varkappa(x)$ in each $\overline{U}_j$ is less than $1/2$. Let $\{\alpha_j(x)\}$ be a partition of unity corresponding to the covering $\{U_j\}$, and let $\varkappa_j = \varkappa(x_j)$, where x_j is an arbitrary point of U_j. By $H_{(\varkappa)+s}(G)$, where $(\varkappa) = \{\varkappa_j\}$, we denote the space of functions $\varphi(x)$ in G, for

for which the norm is finite

$$\|\varphi\|^2_{(\chi)+s} = \sum_j \|\alpha_j \varphi\|^2_{\chi_j+s}, \tag{3}$$

where $\| \; \|_{\chi_j+s}$ is the Sobolev-Slobodetskii norm. The space obtained by closing the finite functions in G in the metric (3) will be denoted by $\mathring{H}_{(\chi)+s}$.

By the first homogeneous boundary-value problem we mean finding a solution of equation (1) from $\mathring{H}_{(\chi)}$. We note that on those parts of the boundary $\Gamma_j = \Gamma \cap \overline{U}_j$, where χ_j is positive, $\varphi(x)$ from $\overset{\circ}{H}_{(\chi)}$ vanishes together with derivatives up to a certain order.

The operator K_α maps $\overset{\circ}{H}_{(\chi)}$ continuously into $H_{(\chi)-\alpha}$. We impose on the operator T the condition that it map $\overset{\circ}{H}_{(\chi)}$ into $H_{(\chi)-\alpha_1}$, where $\alpha_1 < \alpha$. For example, T satisfies this condition if $T = \sum_i K_{\beta_i}$, where K_{β_i} are operators of the form K_α with $\beta_i < \alpha$.

Theorem 1. *The operator $K = K_\alpha + T$, under conditions a) and b), is a Φ-operator $*$ from $\overset{\circ}{H}_{(\chi)}$ into $H_{(\chi)-\alpha}$.*

It follows from this that under these conditions equation (1) is normally solvable and the estimate holds

$$\|\varphi\|_{(\chi)} \leq C\left(\|f\|_{(\chi)-\alpha} + \|\varphi\|_{(\chi)-1}\right), \qquad \varphi \in \overset{\circ}{H}_{(\chi)}. \tag{4}$$

We note that in the choice of the space $\overset{\circ}{H}_{(\chi)}$ there is the arbitrariness indicated above. However, if in at least one boundary neighborhood U_j one chooses $\hat{\chi}_j$, differing from $\chi(x_j) = \chi_j$, where $x_j \in \overline{U}_j \cap \Gamma$, by more than 1/2, then the operator K will not be a Φ-operator from $\overset{\circ}{H}_{(\chi)}$ into $H_{(\chi)-\alpha}$ when (χ) contains the component $\hat{\chi}_j$ instead of χ_j. In particular, from Theorem 1 there follows the normal solvability in the spaces $\overset{\circ}{H}_{(\chi)}$ of the singular integral equation (1) in the domain G. We draw attention to the fact that if $\max_{x\in\Gamma} |\chi(x)| \geq 1/2$, then the operator (1) in the space $L_2(G)$, equivalent to $H_{(0)}$, will not be a Φ-operator.

The main point of the proof is the study of the equation $K_\alpha\varphi = f$ with symbol $\widetilde{K}_\alpha$, independent of x, in the half-space $x_n > 0$. Then the problem is solved in explicit form by the Wiener-Hopf method (see ([3])).

Theorem 1 is also true in the spaces $H_{(\chi),N}(G)$ of functions smooth inside the domain G, with norm

$$\|\varphi\|^2_{(\chi),N} = \sum_j \|\alpha_j\varphi\|^2_{\chi_j,N}, \quad \text{where } \|\varphi\|^2_{\chi_j,N} = \sum_{k=0}^{N} \|\beta_k\varphi\|^2_{\chi_j+k},$$

$\beta_k(x)$ is a smooth function in $G \cup \Gamma$ such that $\beta_k(x) = O(r^k)$, where r is the distance from x to Γ; $\beta_k(x) > 0$ for $x \in G$.

By the nonhomogeneous first boundary-value problem we mean the problem of solving equation (1), in which the integrals are taken over R^n and for $x \in R^n \setminus G$ the sought solution is prescribed as $\varphi(x) = f(x)$ $(x \in R^n \setminus G)$.

Theorem 2. *The nonhomogeneous first boundary-value problem is normally solvable, i.e. the operator $K\varphi = (F, f(x))$ is a Φ-operator from $H_{(\chi)}(R^n)$ into $(H_{(\chi)-\alpha}(G), H_{(\chi)}(R^n \setminus G))$.*

* Definition of a Φ-operator, see in ([6]).

3. Let us consider general boundary-value problems for equation (1). In addition to conditions a) and b), we impose on $\widetilde{K}_\alpha(x,\xi)$ the following additional

Condition c). $\varkappa = m$ is an integer and is the same for all $x \in \Gamma$, and for any integer $p \geq -m$

$$(\xi_n - i|\xi'|)^p K_m^+(x_0,\xi',\xi_n) = P_{m+p}(x_0,\xi',\xi_n) + R_{m+p}(x_0,\xi',\xi_n), \tag{5}$$

where P_{m+p} is a polynomial in ξ_n of degree $m+p$, and

$$|R_{m+p}(x_0,\xi',\xi_n)| \leq \frac{C|\xi'|^{m+p+1}}{|\xi'|+|\xi_n|}; \tag{6}$$

P_{m+p} and R_{m+p} are homogeneous functions of $\xi = (\xi',\xi_n)$ of degree $m+p$.

Condition c) can be replaced by the somewhat less restrictive, but more convenient, condition c′).

Condition c′). The kernel $K_m^+(x_0,\xi',\xi_n)$ admits analytic continuation in ξ_n into a part of the half-plane $\operatorname{Im}\xi_n < 0$, namely for $|\xi_n| > M|\xi'|$, and the resulting function is single-valued and analytic outside the semicircle $C_0: \ |\xi_n| \leq M|\xi'|$, $\operatorname{Im}\xi_n < 0$. This condition is, obviously, always satisfied by operators K_α for which $\widetilde{K}_\alpha(x,\xi)$ is rational in ξ_n.

If condition c) or c′) is fulfilled, then for any s the operator K_α maps $H^s(G)$ into $H^{s-\alpha}(G)$.

We consider separately two cases: 1) $m \geq 0$ and 2) $m < 0$.

1) $m \geq 0$. In this case, together with equation (1) in G, m boundary conditions are prescribed on Γ:

$$B_j\varphi|_\Gamma = g_j(x_1), \qquad x_1 \in \Gamma \qquad (j = 1,\dots,m), \tag{7}$$

where $B_j = B_{\alpha_j} + T^{(j)}$ are general operators of the form (1); B_{α_j} has order of homogeneity α_j, where α_j is an arbitrary real number; $T^{(j)}$ is an operator subordinate to B_{α_j}. Note that the number m of boundary conditions (7) is determined by the index m of the kernel K_m^+ in (2) ($\varkappa = m$), and not by the order of homogeneity α of the whole operator K_α. For normal solvability of problem (1), (7), it is necessary to impose on B_j the following regularity condition (an analogue of the Shapiro–Lopatinskii condition):

$$\det \left\| \int_{\Gamma_0} \frac{\widetilde{B}_{\alpha_j}(x_0, \xi', \xi_n)\, \xi_n^{k-1}}{K_m^+(x_0, \xi', \xi_n)}\, d\xi_n \right\| \neq 0 \qquad \text{for } \xi' \neq 0,\ 1 \leq j, k \leq m, \tag{8}$$

where Γ_0 is the boundary of the semicircle C_0. It is assumed here that B_{α_j} and K_m^+ satisfy condition c′). The regularity condition is formulated analogously when condition c) is fulfilled.

Theorem 3. Suppose that in G equation (1) is given and on Γ the boundary conditions (7) are given, where K_m^+ satisfies conditions a), b), c) or c′), and B_{α_j} satisfies conditions a), c′) and (8). Suppose, further, that T acts from $H^s(G)$ to $H^{s-\alpha-\delta}(G)$, and $T^{(j)}$ acts from $H^s(G)$ to $H^{s-\alpha_j-\delta}(G)$, $\delta > 0$. Then problem (1), (7) is normally solvable in the space $H^s(G)$ (where $s \geq \alpha_j + 1$) and the estimate

$$\|\varphi\|_s \leq C \left(\|F\|_{s-\alpha} + \sum_{j=1}^{m} \|g_j\|_{s-\alpha_j-1/2} + \|\varphi\|_{s-1} \right) \tag{9}$$

holds.

This theorem generalizes to the case of very general operators the well-known theory of elliptic differential boundary-value problems (see, for example, [7–9])*.

* The case in which $\widetilde{K}_\alpha(x, \xi', \xi_n)$ and $\widetilde{B}_{\alpha_j}(x, \xi', \xi_n)$ for $x \in \Gamma$ depend polynomially on ξ_n was previously considered by other methods by M. S. Agranovich.

Generalization. The theory of boundary-value problems of the form (1), (7) also extends to the case when the number $\varkappa = \varkappa(x)$ in the factorization (2) is not an integer and depends on x.

2) $m < 0$. In this case it is natural, instead of equation (1), to consider the more general equation

$$\int_G L_\alpha(x, x-y) F(y)\, dy + \sum_{j=1}^{|m|} \int_\Gamma G_j(x, x-y_1) g_j(y_1)\, dy_1 + \cdots = \varphi(x), \tag{10}$$

where the ellipsis denotes two subordinate terms of analogous form. The kernels $L_\alpha(x, z)$ and $G_j(x, z)$ are, generally speaking, generalized functions in z. The kernel $L_\alpha(x, z)$ satisfies conditions a), b), c) or c′), where $m < 0$. The kernels $\widetilde{G}_j(x, \xi', \xi_n)$, $j = 1, \dots, |m|$, have order of homogeneity m_j in ξ and satisfy condition c) (or c′)). Thus, in view of the fact that $m < 0$, in equation (10) there appear $|m|$ additional integrals of potential type, instead of the m boundary conditions (7) encountered earlier for $m > 0$.

Let us denote:

$$R_j^+(x_0,\xi',\xi_n) = \lim_{\varepsilon\to+0}\int_{\Gamma_0} \frac{\widetilde{G}_j(x_0,\xi',\eta_n)\,d\eta_n}{L^-_{-|m|+\alpha}(x_0,\xi',\eta_n)(\eta_n-(\xi_n+i\varepsilon))}, \tag{11}$$

where $L^-_{-|m|+\alpha}(x_0,\xi',\xi_n)$ enters the following factorization of L_α:

$$L_\alpha(x_0,\xi',\xi_n) = \frac{L^-_{-|m|+\alpha}(x_0,\xi',\xi_n)}{L^+_{|m|}(x_0,\xi',\xi_n)}.$$

It is proved that R_j^+ satisfies condition c$'$) and, consequently,

$$L^+_{|m|}(x_0,\xi',\xi_n)R_j^+(x_0,\xi',\xi_n) = P_j(x_0,\xi',\xi_n) + R_{j1}(x_0,\xi',\xi_n), \tag{12}$$

where P_j are polynomials in ξ_n of degree $\leq |m|-1$; for R_{j1} an estimate of the form (6) holds.

The regularity condition for equation (10) consists in the polynomials P_j ($j = 1,\dots,|m|$) being linearly independent for any fixed $|\xi'| \neq 0$ and $x_0 \in \Gamma$.

Theorem 4. *Let the kernel L_α satisfy conditions a), b), c$'$) or c), the kernels G_j satisfy conditions c$'$) or c) and a) (with order of homogeneity m_j), and let the regularity condition be fulfilled. Then equation (10) is normally solvable if the unknown functions $(F(x), g_k(x_1))$ $(k = 1,\dots,|m|,\ x_1 \in \Gamma)$ belong to the spaces $(H^s(G), H^{s-|\alpha|+m_k-1/2}(\Gamma))$, while the prescribed function $\varphi(x)$ belongs to $H^{s-\alpha}(G)$. In this case the estimate*

$$\|F\|_s + \sum_{k=1}^{|m|}\|g_k\|_{\lambda_k} \leq C\left(\|\varphi\|_{s-\alpha} + \|F\|_{s-1} + \sum_{k=1}^{|m|}\|g_k\|_{\lambda_k-1}\right),$$

holds, where $\lambda_k = s-\alpha+m_k+\frac{1}{2}$.

By analogous methods we have also studied systems of singular equations and applications to boundary-value problems with discontinuous boundary conditions.

In conclusion, we note that all our results have been carried over to the case of singular equations of “parabolic”type. In this case general mixed boundary-value problems have been studied.

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Note: Figure translations are in progress. See original paper for figures.