ON THE QUESTION OF THE EXISTENCE OF SIMILAR TESTS FOR THE BEHRENS-FISHER PROBLEM

1964

SovietRxiv

Abstract

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MATHEMATICS

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ON THE QUESTION OF THE EXISTENCE OF SIMILAR TESTS FOR THE BEHRENS-FISHER PROBLEM

(Presented by Academician V. I. Smirnov on 10 X 1963)

Let $x_1, \dots, x_n \in N(a_1, \sigma_1)$ and $y_1, \dots, y_m \in N(a_2, \sigma_2)$ be two independent repeated normal samples with four unknown parameters $a_1, a_2, \sigma_1, \sigma_2$. The Behrens–Fisher problem consists in describing all tests similar with respect to σ_1, σ_2 and $a = a_1 = a_2$. We note that, in the present case, every test that is not similar is biased.

The presence of the relation $a_1 = a_2$ takes this problem beyond the framework of classical theory (^{1}), raising the natural question of the very existence of nontrivial similar tests of various kinds and classes.

The general problem of existence is solved comparatively simply by means of "studentization" * of the problem. This method makes it possible to construct a whole class of nonrandomized similar tests (^{2}). In (^{3}) a class of randomized similar tests is constructed that depend only on the sufficient statistics

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \bar{y} = \frac{1}{m}\sum_{j=1}^{m} y_j, \qquad s_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2, \qquad s_2^2 = \frac{1}{m}\sum_{j=1}^{m}(y_j - \bar{y})^2.$$

Since the "studentization" of the problem does not give tests with this property, the question of the existence of nonrandomized similar tests depending only on the indicated sufficient statistics acquires independent significance. Up to now no exhaustive answer has been given to it.

Below we study the Behrens-Fisher problem in the class of homogeneous nonrandomized tests, i.e., those that are associated with critical regions located in the half-strip $\Omega = (-\infty < \xi < \infty;\ 0 \le \eta < \infty)$, $\xi = (\bar{x} - \bar{y})/s_2$, $\eta = s_1/s_2$, where the samples induce the family of densities

$$C_{m,N}\vartheta^{m/2}(1+\vartheta)^{-N-1/2}\frac{\eta^{2N-m-1}}{[(\vartheta-\vartheta_1)(\vartheta-\vartheta_2)]^N},$$

where $N = (m+n-1)/2$, $\vartheta = m\sigma_1^2/n\sigma_2^2$, and ϑ_1, ϑ_2 are the roots of the equation

$$\vartheta^2 + \vartheta(1+\xi^2+\eta^2) + \eta^2 = 0, \qquad -1 \le \vartheta_1 \le 0, \quad -\infty < \vartheta_2 \le -1.$$

Zones of this type were introduced in (^{4}) on the basis of A. Wald' s axioms. Obviously, a similar test from our class is described by a function taking the two values 0 and 1, the distribution of which does not depend on ϑ.

Let, more generally, $g(\xi,\eta)$ be a Lebesgue-measurable function whose distribution does not depend on ϑ, i.e.

$$\iint\limits_{\Omega} \Psi[g(\xi,\eta)] \frac{\eta^{2N-m-1}\, d\xi\, d\eta}{[(\vartheta-\vartheta_1)(\vartheta-\vartheta_2)]^N} = C_\Psi \vartheta^{-m/2}(1+\vartheta)^{-(N-1/2)}, \tag{1}$$

where Ψ is an arbitrary Borel-measurable bounded function, $\Psi(0)=0$, and C_Ψ is a constant depending on Ψ.

In previous papers (^{4,5}) the concept of a singularly varying test was introduced, reflecting the idea that certain features of the behavior of $g(\xi,\eta)$ on the whole half-strip Ω must be observed in its finite part, provided only that $g(\xi,\eta)$ satisfies (1). It was found that such a function $g(\xi,\eta)$ already in the finite part of Ω, roughly speaking, attains both its minimum and its maximum. The theorem stated below strengthens this result still further, referring the singularity of the variation to a quite definite point.

Theorem. *If* $m = 4s+3,\ n = 2r$ *or* $m = 2r,\ n = 4s+3$, *and the Borel set* M, *not containing zero, is such that* $\operatorname{mes} g^{-1}(M) > 0$, *while* $g(\xi,\eta)$ *satisfies (1), then* $\operatorname{mes}\{\Delta \cap g^{-1}(M)\} > 0$, *whatever the neighborhood* Δ *of the point* $(\xi=0, \eta=1)$.

In other words, the entire wealth of values of $g(\xi,\eta)$ on the half-plane Ω must be exhausted in an arbitrarily small neighborhood of the point $(0,1)$, if the function $g(\xi,\eta)$ is a solution of (1).

The proof of the theorem is as follows. We note that in (1) one may always replace m by n, passing to the new variables $\xi^* = \xi/\eta$, $\eta^* = 1/\eta$ and taking $1/\vartheta$ instead of ϑ. Let now $m = 4s+3,\ n = 2r$ and $\operatorname{mes}\{\Delta \cap g^{-1}(M)\} = 0$. We may assume that the intersection $\Delta \cap g^{-1}(M)$ is empty. Put

$$\Psi(z) = \begin{cases} 1, & z \in M, \\ 0, & z \notin M, \end{cases}$$

and in (1) pass from (ξ,η) first to the variables $(\vartheta_1,\vartheta_2)$, and then to the variables $x = -\vartheta_1,\ y = -\vartheta_2$. We obtain

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* By this is meant reduction to the case of unequal variances.

$$\iint_R \left[\chi_{g^{-1}(M)}(\xi,\eta)+\chi_{g^{-1}(M)}(-\xi,\eta)\right]\frac{(y-x)(xy)^{N-m/2-1}\,dx\,dy}{\sqrt{1-x}\sqrt{y-1}\,[(\vartheta+x)(\vartheta+y)]^N}=\frac{C}{\vartheta^{m/2}(1+\vartheta)^{N-1/2}}, \tag{2}$$

where $R=(0\le x\le 1;\ 1\le y<\infty)$, $\chi_{g^{-1}(M)}(\xi,\eta)$ is the characteristic function of the set $g^{-1}(M)$, and $C>0$. It is easy to verify that $\chi_{g^{-1}(M)}(\xi,\eta)$ and $\chi_{g^{-1}(M)}(-\xi,\eta)$ vanish in some small neighborhood of the point $(x=1,y=1)$; this guarantees the summability in R of the functions

$$\left[\chi_{g^{-1}(M)}(\xi,\eta)+\chi_{g^{-1}(M)}(-\xi,\eta)\right]\frac{(xy)^{N-m/2-1}}{\sqrt{1-x}\sqrt{y-1}(y-x)^{2N-k-1}},$$

$k=1,\dots,N$, and the possibility of the interchanges of the limits of integration needed below.

Decomposing

$$\frac{1}{[(\vartheta+x)(\vartheta+y)]^N}$$

into elementary fractions and denoting by $\varphi_k(x)$ and $\psi_k(y)$, $k=1,\dots,N$, respectively the integrals

$$\int_1^\infty \left[\chi_{g^{-1}(M)}(\xi,\eta)+\chi_{g^{-1}(M)}(-\xi,\eta)\right]\frac{(xy)^{N-m/2-1}\,dy}{\sqrt{1-x}\sqrt{y-1}(y-x)^{2N-k-1}},$$

$$\int_0^1 \left[\chi_{g^{-1}(M)}(\xi,\eta)+\chi_{g^{-1}(M)}(-\xi,\eta)\right]\frac{(xy)^{N-m/2-1}\,dx}{\sqrt{1-x}\sqrt{y-1}(y-x)^{2N-k-1}},$$

we find from (2)

$$\sum_{k=1}^N(-1)^{N-k}C_{2N-k-1}^{N-1}\int_0^1\frac{\varphi_k(x)\,dx}{(\vartheta+x)^k}+\sum_{k=1}^N(-1)^N C_{2N-k-1}^{N-1}\int_1^\infty\frac{\psi_k(y)\,dy}{(\vartheta+y)^k}=\frac{C}{\vartheta^{m/2}(1+\vartheta)^{N-1/2}}.$$

Inversion of the Laplace transform, divided by t^{N-1}, gives

$$\sum_{k=1}^N(-1)^{N-k}\frac{C_{2N-k-1}^{N-1}}{\Gamma(k)}\int_0^1\varphi_k(x)\frac{e^{-xt}}{t^{N-k}}\,dx+\sum_{k=1}^N(-1)^N\frac{C_{2N-k-1}^{N-1}}{\Gamma(k)}\int_1^\infty\psi_k(y)\frac{e^{-yt}}{t^{N-k}}\,dy=$$

$$= \frac{C}{\Gamma(m/2)\Gamma(N-1/2)}\, t^{(m-1)/2} \int_0^1 \tau^{N-3/2}(1-\tau)^{m/2-1} e^{-t\tau}\, d\tau. \qquad (3)$$

Introduce the functions

$$f_k(\tau, z) = \begin{cases} \dfrac{(\tau - z)^{N-k-1}}{\Gamma(N-k)}, & \tau > z, \\ 0, & 0 < \tau < z, \end{cases} \qquad k = 1, \dots, N-1.$$

Then the left-hand side of (3) will be the Laplace integral of the function

$$\begin{aligned} h(\tau) &= \sum_{k=1}^{N-1} (-1)^{N-k} \frac{C_{2N-k-1}^{N-1}}{\Gamma(k)} \int_0^1 \varphi_k(x) f_k(\tau, x)\, dx \\ &\quad + \sum_{k=1}^{N-1} (-1)^{N} \frac{C_{2N-k-1}^{N-1}}{\Gamma(k)} \int_1^\infty \psi_k(y) f_k(\tau, y)\, dy \\ &\quad + \begin{cases} \dfrac{1}{\Gamma(N)} \varphi_N(\tau), & 0 < \tau < 1, \\ (-1)^N \dfrac{1}{\Gamma(N)} \psi_N(\tau), & 1 < \tau, \end{cases} \end{aligned}$$

and the right-hand side—the Laplace integral of the function

$$h^*(\tau) = \begin{cases} \dfrac{C}{\Gamma(m/2)\Gamma(N-1/2)} \dfrac{d^{(m-1)/2}}{d\tau^{(m-1)/2}} \left[\tau^{N-3/2}(1-\tau)^{m/2-1}\right], & 0 < \tau < 1, \\ 0, & 1 < \tau. \end{cases}$$

Compare $h(\tau)$ and $h^*(\tau)$ as $\tau \to 1-$. In the interval $0 < \tau < 1$, $h(\tau)$ is bounded below. However, as one approaches the point 1 from the left, $h^*(\tau)$ has order $(1-\tau)^{-1/2}$, and since $(-1)^{(m-1)/2} = -1$, $h^*(\tau)$ in the interval $0 < \tau < 1$ is not bounded below. This contradiction proves the theorem.

Consequence. In the case when either $m = 4s+3$, $n = 2r$, or $m = 2r$, $n = 4s+3$, there can exist no zonal statistic for the Behrens–Fisher problem that is continuous at the point $(0, 1)$. Moreover, the theorem shows that any similar homogeneous nonrandomized test cannot be regarded as satisfactory. However small the difference $\bar{x} - \bar{y}$ may be when s_1 and s_2 are not very small and not very large, it must reject the null hypothesis with positive probability; meanwhile, the smallness of $\bar{x} - \bar{y}$ under these conditions points to the truth of the hypotheses of equality of means.

The proof method given for the theorem can also be applied to the study of Wald tests [6] with a good boundary. Namely, it can be proved that, except for the identically zero function, there exists no Wald function having derivatives of sufficiently high order at a single unique point $\eta = 1$.

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Received
21 IX 1963

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Note: Figure translations are in progress. See original paper for figures.