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Abstract

Full Text

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ON THE DEVIATION OF ANALYTIC FUNCTIONS FROM THE ARITHMETIC MEANS OF PARTIAL SUMS OF A FABER SERIES

(Presented by Academician S. N. Bernstein, 10 II 1964)

In considering methods of approximation of periodic functions $f(x)$ of period 2π, it is of interest to obtain an estimate of the approximation of a function in dependence on the behavior of the sequence $E_n(f)$ of best approximations of the function f.

In the case of uniform approximation of a continuous 2π-periodic function by Fejér sums, S. B. Stechkin ([1]) and M. F. Timan ([2]), by different methods, obtained the estimate

$$|f(x) - \sigma_n(f)| \leq \frac{C_1}{n} \sum_{\nu=1}^{n} E_\nu(f), \tag{1}$$

where $\sigma_n(f)$ is the arithmetic mean of the partial sums of the Fourier series of the function $f(x)$, and C_1 is an absolute constant.

Let now $f(x)$ be a continuous function defined on a finite interval. In this case, as is known (see ([3])), instead of uniform approximations by algebraic polynomials it is more natural to consider approximations taking into account the position of the point x on this interval. This is seen from the well-known theorem of A. F. Timan (([4]), see also ([3]), 5.2), which asserts that for any function $f(x)$ which has on $[-1, 1]$ a continuous derivative of order r, there exists a sequence of algebraic polynomials $P_n(x)$ of degree $\leq n$, $n > r$, satisfying for all $x \in [-1, 1]$ the inequality:

$$|f(x) - P_n(x)| \leq \frac{C_r}{n^r} \left(\sqrt{1-x^2} + \frac{1}{n}\right)^r \omega_r \left\{\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right\}, \tag{2}$$

where $\omega_r(t)$ is the modulus of continuity of the r-th derivative of $f(x)$.

The right-hand side of inequality (2) always represents a certain function

$$\omega\left\{\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right\},$$

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where $\omega(t)$ has the following properties: $\omega(0) = 0$, $\omega(t)$ does not decrease together with t, and

$$\omega(t_1 + t_2) \leq M\{\omega(t_1) + \omega(t_2)\},$$

where M is some constant.

In connection with this, for an arbitrary function $\omega(t)$ possessing the indicated properties, it is natural to introduce into consideration the class $A_\omega[-1, 1]$ of all functions $f(x)$ defined on $[-1, 1]$ for which, for every $n = 1, 2, 3, ...$, there exists an algebraic polynomial $P_n(x)$ of degree $\leq n$ such that

$$|f(x) - P_n(x)| \leq \omega\left\{\frac{1}{n}\left(\sqrt{1-x^2} + \frac{1}{n}\right)\right\}. \tag{3}$$

We shall approximate functions $f(x) \in A_\omega[-1, 1]$ by arithmetic means of the partial sums of their P. L. Chebyshev series.

Let

$$\widehat{T}_0(x) = \sqrt{\frac{1}{\pi}}, \qquad \widehat{T}_k(x) = \sqrt{\frac{2}{\pi}} \cos k \arccos x \quad (k = 1, 2, 3, ...) \tag{4}$$

be the system of P. L. Chebyshev polynomials orthonormal on $[-1, 1]$ with weight

$$\frac{1}{\sqrt{1-x^2}}$$

.

were,

$$C_k = \int_{-1}^{1} \frac{f(t)\, \hat{T}_k(t)}{\sqrt{1-t^2}}\, dt \tag{5}$$

are the Fourier coefficients of the function $f(x)$ with respect to this system, and

$$\sigma_n(f; x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) C_k \hat{T}_k(x) \tag{6}$$

is the Fejér sum with respect to the polynomials of P. L. Chebyshev.

The following theorem gives an estimate of the quantity

$$R_n(f, x) = |\, f(x) - \sigma_n(f, x)\,| \tag{7}$$

for functions $f(x) \in A_\omega[-1, 1]$ at each point x of the interval $[-1, 1]$.

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Theorem 1. If $f(x) \in A_\omega[-1,1]$, then for all $x \in [-1,1]$, for any $n = 1, 2, 3, ...$, the inequality

$$|\, f(x) - \sigma_n(f,x)\,| \leq \frac{C_2}{n} \sum_{k=1}^{n} \omega \left\{ \frac{1}{k} \left(\sqrt{1-x^2} + \frac{1}{k} \right) \right\}, \tag{8}$$

holds, where C_2 is a certain constant independent of f, x, and n.

Inequality (8) is sharp in the sense of order. For any point $x \in [-1,1]$ one can indicate a function $f(t) \in A_\omega[-1,1]$ such that, for all $n = 1, 2, 3, ...$,

$$|\, f(x) - \sigma_n(f,x)\,| \geq \frac{C_3}{n} \sum_{k=1}^{n} \omega \left\{ \frac{1}{k} \left(\sqrt{1-x^2} + \frac{1}{k} \right) \right\}, \tag{9}$$

where C_3 is a positive constant independent of n.

Theorem 1 gives an estimate of the quantity (7) that takes into account the position of the point x on $[-1,1]$ and depends on the constructive properties of the function $f(x) \in A_\omega[-1,1]$. One can give an estimate of the quantity (7) for any continuous function $f(x)$ defined on $[-1,1]$, depending on its structural properties and still taking into account the position of the point on the interval.

The following proposition holds.

If $f(x) \in C_{[-1,1]}$, then

$$|f(x) - \sigma_n(f,x)| \leq \frac{1}{n} \sum_{k=1}^{n} \left\{ \omega_2 \left(f; \frac{\sqrt{1-x^2}}{k} \right) + \omega \left(f; \frac{1}{k^2} \right) \right\},$$

where $\omega(f;\delta)$ and $\omega_2(f;\delta)$ are the moduli of smoothness of the function $f(x)$ of the first and second orders, respectively.

If the interval $[-1,1]$ is considered in the complex plane, then the polynomials of P. L. Chebyshev

$$\frac{1}{2^{k-1}} \cos k \arccos z,$$

which deviate least from zero, are the Faber polynomials constructed by means of the function

$$\varphi(z) = \frac{1}{2} \left(z + \sqrt{z^2 - 1} \right),$$

which conformally maps the exterior of the interval $[-1,1]$ onto the domain $|\varphi(z)| > \frac{1}{2}$ (see (5)).

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Let now G be an arbitrary domain with a simply connected complement and boundary C. By means of the function $w = \varphi(z)$ we conformally map the exterior of the domain G onto the domain $|w| > R$ so that the conditions

$$\varphi(\infty) = \infty; \qquad \lim_{z\to\infty} \frac{\varphi(z)}{z} = 1.$$

are satisfied.

Let us denote by $\psi(w)$ the function inverse to $w = \varphi(z)$. We shall assume that it is continuous in $|w| \geq R$. Let $\Phi_n(z)$ $(n = 0, 1, 2, ...)$ be the corresponding system of Faber polynomials for the domain G.

We now consider a function $f(z)$, analytic in G and continuous in $\overline{G}$. For this function form the sum

$$\sigma_n(f; \Phi_n; z) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k \Phi_k(z),$$

where

$$a_k = \frac{1}{2\pi i} \int_{|w|=R} f[\psi(w)] \frac{dw}{w^{k+1}},$$

which is the arithmetic mean of the partial sums of the Faber series, and consider the quantity

$$|f(z) - \sigma_n(f; \Phi_n; z)|. \tag{10}$$

The maximum of this function is attained on the boundary of the domain, and we shall estimate the magnitude of this deviation depending on the position of the point z on the boundary of the domain G.

If G is the unit disk, $\varphi(z) = z$, then we have

$$\sigma_n(f; \Phi_n; z) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k z^k,$$

where a_k are the Taylor coefficients of the function $f(z)$. In this case one can obtain the following result:

$$|f(z) - \sigma_n(f; \Phi_n; z)| \leq \frac{C_4}{n} \sum_{k=0}^{n} E_k(f),$$

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where $E_k(f)$ is the sequence of best approximations to the function f on the boundary of the unit disk by polynomials of the form $\sum_{k=0}^{n} C_k z^k$; C_4 is an absolute constant.

For the domain G, which is a disk, a uniform estimate over all z lying on the boundary is natural.

If certain restrictions are imposed on the domain G, then the estimate of the quantity (10) can be expressed in terms of the structural properties of the function $f(z)$ on the boundary of the domain G.

Theorem 2. Let G be an arbitrary finite domain with simply connected complement, whose boundary C consists of a finite number of arcs $C^{(j)}$ $(j = 1, 2, \dots, k)$ with continuous curvature, forming at their junction points z_j angles $\alpha_j \pi$, $0 < \alpha \leq 3/2$, and possessing the property that in a neighborhood of each of the junction points z_j the function $w = \varphi(z)$ can be represented in the form

$$\varphi(z) = \lambda(z)(z - z_j)^{1/(2-\alpha_j)} + \varphi(z_j),$$

where $\lambda(z)$ is a function continuous in some neighborhood of the point z_j together with its first and second derivatives $\lambda'(z)$, $\lambda''(z)$, and $\lambda(z_j) \neq 0$.

Suppose, moreover, that $f(z)$ is a function analytic in G and continuous in $\overline{G}$, with modulus of continuity $\omega(f; t)$ on the boundary of the domain. Then

$$|f(z) - \sigma_n(f; \Phi_n; z)| \leq \frac{C_5}{n} \sum_{k=1}^{n} \omega\left[f; \rho_{1+1/k}(z)\right],$$

where $\rho_{1+1/k}(z)$ is the distance from the point z on the boundary of the domain G to the level line $|\varphi(z)| = \left(1 + \frac{1}{k}\right) R$, and C_5 is a constant independent of f, n, z.

Domains satisfying the conditions of Theorem 2 were considered by V. K. Dzyadyk [(6)].

In conclusion I express my deep gratitude to A. F. Timan for posing the problem and for his attention to this work.

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