Mathematics

V. V. PROIZVOLOV

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Abstract

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V. V. PROIZVOLOV

ON DIVIDING MAPPINGS OF LOCALLY CONNECTED SPACES

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A mapping f of a space X into a space Y is called **dividing** if, for any point $x \in X$ and any neighborhood Ox of it, there exists a neighborhood Oy of the point $y = fx \in Y$ such that its complete preimage is divided into the sum of two disjoint open sets G and H in such a way that $x \in G \subseteq Ox$. This concept, introduced by A. Zarelua ([1]), has proved useful in general dimension theory.

All mappings in this paper are continuous, and all spaces are Hausdorff; by a zero-dimensional mapping we mean here a mapping under which the preimage of every point has dimension zero ind.

Theorem 1. *Let there be a dividing mapping $f : X \to Y$, where X is a locally connected space and Y is metrizable. Then the space X is metrizable.*

Proof. The metric space Y, by Bing's metrization criterion ([2]), has a σ-discrete base $\gamma = \{\gamma_i\}$, where each γ_i is a discrete system of open sets: $\gamma_i = \{U_{i\alpha}\}$; $i = 1, 2, \ldots$. A system of open sets in a space is called discrete if every point of the space has a neighborhood meeting only finitely many elements of this system.

Let $V_{i\alpha}$ denote the $1/i$-neighborhood of the set $U_{i\alpha}$, i.e., the set of all points whose distance from the points of $U_{i\alpha}$ is no more than $1/i$. The collection of all $V_{i\alpha}$ (over all i and all α) forms a base of the space Y; denote it by $\omega = \{V_{i\alpha}\}$.

Observe that every open subset of a locally connected space decomposes into open components. Hence

$$f^{-1}V_{i\alpha} = \bigcup_{\delta} W_{i\alpha\delta},$$

where each $W_{i\alpha\delta}$ is open and connected, and

$$W_{i\alpha\delta_1} \cap W_{i\alpha\delta_2} = \Lambda, \qquad \delta_1 \neq \delta_2.$$

The union of all $W_{i\alpha\delta}$ over all indices i, α, δ is a system of open subsets of the space X; denote it by W. We shall verify that $W = \{W_{i\alpha\delta}\}$ is a base in X.

Take an arbitrary point $x \in X$ with an arbitrary neighborhood Ox of it. We must show that there is $W_{i\alpha\delta} \in W$ such that $x \in W_{i\alpha\delta}$ and $W_{i\alpha\delta} \subseteq Ox$. Since the mapping f is dividing, there exists an element $V_{i\alpha} \in \omega$ such that $fx \in V_{i\alpha}$ and

$$f^{-1}V_{i\alpha} = O_1 \cup O_2,$$

where O_1 and O_2 are open in X, with

$$O_1 \subseteq Ox, \qquad O_1 \cap O_2 = \Lambda.$$

The set O_1 has a component $W_{i\alpha\delta}$ such that $W_{i\alpha\delta} \ni x$, and thereby it is proved that W is a base in X.

Similarly one checks that the system of open sets

$$Z = \{Z_{i\alpha\delta}\},$$

where

$$Z_{i\alpha\delta} = W_{i\alpha\delta} \cap f^{-1}U_{i\alpha},$$

is also a base in X.

We shall prove that the base $Z = \{Z_{i\alpha\delta}\}$ is σ-discrete. By Bing' s criterion, this will prove the metrizability of the space X.

Fix i and consider the system

$$Z_i = \{Z_{i\alpha\delta}\};$$

we shall prove that Z_i is a discrete system of sets in X. First of all, this system is disjoint. Suppose for a moment that it is not discrete, i.e., suppose that there is a point $x \in X$ such that every neighborhood of it meets infinitely many elements of the system $Z_i = \{Z_{i\alpha\delta}\}$. Take such a neighborhood O_y of the point y

where $y = fx$, that from the fact that Oy intersects an element $U_{i\alpha} \in \gamma_i$ it follows that $y \in [U_{i\alpha}]$, and there are only finitely many such elements with which Oy intersects: $U_{i\alpha_1}, U_{i\alpha_2}, \dots, U_{i\alpha_m}$. And one more requirement for Oy: it is necessary that

$$Oy \subseteq \bigcap_{k=1}^{m} V_{i\alpha_k}.$$

Now take that component of the set $f^{-1}Oy$ which contains the point x, and denote it by Ox. For each α_k, $k = 1, 2, \dots, m$, there is a δ_k such that $Ox \subseteq W_{i\alpha_k\delta_k}$. It is not difficult to verify that, in order that $Ox \cap Z_{i\alpha\delta} \neq \Lambda$, where $Z_{i\alpha\delta} \in Z_i$, it is necessary that the index α be equal to one of the α_k, and the index δ be equal to one of the δ_k, $k = 1, 2, \dots, m$. Indeed, if $Ox \cap Z_{i\alpha\delta} \neq \Lambda$ and $\alpha \neq \alpha_k$, $k = 1, 2, \dots, m$, then Oy would not intersect $U_{i\alpha}$, which would mean that $Ox \cap Z_{i\alpha\delta} = \Lambda$, since $Z_{i\alpha\delta} = W_{i\alpha\delta} \cap f^{-1}U_{i\alpha}$, while the neighborhood $Ox \subseteq f^{-1}Oy$. It is also impossible that $Ox \cap Z_{i\alpha_k\delta} \neq \Lambda$, and $\delta \neq \delta_k$, $k = 1, 2, \dots, m$, since from

$Ox \subseteq W_{i\alpha_k\delta_k}$ it follows that $Ox \cap W_{i\alpha_k\delta} = \Lambda$, if $\delta \neq \delta_k$, $k = 1, 2, \dots, m$, and hence, all the more, $Ox \cap Z_{i\alpha_k\delta} = \Lambda$, since $Z_{i\alpha_k\delta} \subseteq W_{i\alpha_k\delta}$. Consequently, the neighborhood Ox intersects no more than m elements of the system Z_i, namely, it is possible that Ox intersects all or some of the elements $Z_{i\alpha_k\delta_k}$, where $k = 1, 2, \dots, m$. In view of the arbitrariness of the point $x \in X$, this means that the system Z_i is discrete.

Thus, the base $Z = \{Z_{i\alpha\delta}\}$ is σ-discrete, and the metrizability of the space X is proved.

A number of important corollaries will be derived from the theorem proved.

Corollary 1. *Let there be a zero-dimensional mapping $f : X \to Y$, where X is a locally connected and locally bicompact space, and Y is a metric space. Then the space X is metrizable.*

The mapping f is a partitioning mapping, since Yu. M. Smirnov noted ([1]) that a mapping of a locally bicompact space is partitioning when it is zero-dimensional. After what has been said, Corollary 1 follows directly from the theorem.

Since a closed zero-dimensional mapping of a normal space is partitioning ([1]), we have

Corollary 2. *Let there be a closed zero-dimensional mapping $f : X \to Y$, where X is a locally connected normal space, and Y is a metric space. Then X is metrizable.*

Corollary 3. *Let there be a mapping $f : X \to Y$, where X is a locally connected and peripherally bicompact space, and Y is a metric space. Moreover, let f be such that $f^{-1}y$ is a discrete-in-itself set for every point $y \in Y$. Then X is metrizable.*

Lemma. *Let there be a mapping $f : X \to Y$, where X is a peripherally bicompact space, and let f be such that $f^{-1}y$ is a discrete-in-itself set for every point $y \in Y$. Then the mapping f is partitioning.*

For an arbitrary point $x \in X$ and an arbitrary neighborhood Ox of it, find such a neighborhood Oy of the point $y = fx$ that $f^{-1}Oy$ is split into the sum of two disjoint open sets G and H in such a way that $x \in G \subseteq Ox$. For the point x, take such a neighborhood $O'x \subseteq Ox$ with a bicompact boundary that $[O'x] \cap f^{-1}y = x$, i.e., in particular, the boundary Γ of the neighborhood $O'x$ contains no points from $f^{-1}y$. Denote $Oy = Y \setminus f\Gamma$; $G = f^{-1}Oy \cap O'x$, $H = f^{-1}Oy \setminus G$. It is easy to verify that Oy is the required neighborhood of the point y.

Taking the lemma into account, Corollary 3 is easily obtained from the theorem.

A base ω of the space X has pointwise power τ, if τ is the minimal cardinal number having the property that, for every point $x \in X$, the cardinality of the set of elements of the base ω containing the point x is not greater than τ.

The minimum of the pointwise powers over all bases of the space is called the pointwise weight of the space.

Theorem 2. Let there be a dispersing mapping $f : X \to Y$, where X is a locally connected space and Y is a space of point weight τ. Then the point weight of the space X is not greater than τ.

Proof. Denote by $\omega = \{\omega_\alpha\}$ a base of the space Y of point cardinality τ. Since X is locally connected, for every α the set $f^{-1}\omega_\alpha$ "breaks up" in X into open components,

$$f^{-1}\omega_\alpha = \bigcup_\delta V_{\alpha\delta}.$$

The union of all $V_{\alpha\delta}$ over all indices α and δ forms a base of the space X, as was shown in the proof of Theorem 1; denote this base by $W = \{V_{\alpha\delta}\}$.

Let us show that the base W has point cardinality τ, and this will prove everything. We must verify that an arbitrary point $x \in X$ belongs to no more than τ elements of W. Denote by $\omega_y = \{\omega_\alpha\}$ the collection of all elements of the base ω containing the point $y = fx \in Y$; the cardinality of $\omega_y = \{\omega_\alpha\}$, by hypothesis, is not greater than τ. For every element $\omega_\alpha \in \omega_y$ there is only one component $V_{\alpha\delta} \in W$ of the set $f^{-1}\omega_\alpha$ containing the point x, and consequently the number of all elements $V_{\alpha\delta}$ of the base W containing the point x will be not greater than τ. The theorem is proved.

Corollary 1. Let there be a zero-dimensional mapping $f : X \to Y$, where X is a locally bicompact and locally connected space, and Y is a space of point weight τ. Then the point weight of the space X is not greater than τ.

Corollary 2. Let there be a closed zero-dimensional mapping $f : X \to Y$, where X is a locally connected normal space and Y is a space of point weight τ. Then the point weight of the space X is not greater than τ.

Corollary 3. Let there be a mapping $f : X \to Y$, where X is a locally connected and peripherally bicompact space, and Y is a space of point weight τ. Moreover, f is such that $f^{-1}y$ is a discrete-in-itself set for every point $y \in Y$. Then the point weight of the space X is not greater than τ.

These corollaries are derived analogously to the way in which the corollaries of Theorem 1 were derived.

Since the point weight of a bicompactum coincides with its integral weight ([3]), Corollary 1 immediately implies the well-known theorem of Mardešić:

A zero-dimensional mapping of a locally connected bicompactum cannot lower its weight.

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

[1] A. Z. Melua, *DAN*, **144**, No. 4, 713 (1962).
[2] R. H. Bing, *Canad. J. Math.*, **3**, No. 2, 175 (1951).
[3] A. Mishchenko, *DAN*, **144**, No. 5, 985 (1952).

Note: Figure translations are in progress. See original paper for figures.