ON THE SPECTRUM OF A REGULAR QUASI-DIFFERENTIAL OPERATOR

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. V. KUZHEL

ON THE SPECTRUM OF A REGULAR QUASI-DIFFERENTIAL OPERATOR

(Presented by Academician A. A. Dorodnitsyn, 16 I 1964)

In this note a connection is established between the eigenvalues of a (in general non-self-adjoint) regular quasi-differential operator and the zeros of a certain "normalized" Wronskian determinant. In addition, a condition is obtained for the completeness of the system of root subspaces of dissipative ([1]) quasi-differential operators.

1. Consider in the space $L^2(a,b)$ a regular quasi-differential operator A, defined by the self-adjoint quasi-differential expression

$$l(y) = \sum_{k=0}^{n} (-1)^k \left(p_{n-k} y^{(k)}\right)^{(k)}$$

and by the boundary conditions

$$\sum_{k=1}^{2n} a_{jk} y^{[k-1]}(a) + \sum_{k=1}^{2n} b_{jk} y^{[k-1]}(b) = 0 \quad (j = 1, 2, \dots, 2n). \tag{1}$$

The equations of the system (1) are naturally assumed to be linearly independent.

Let r_a and r_b be the ranks, respectively, of the matrices $\|a_{ik}\|$ and $\|b_{ik}\|$. In what follows we consider the case when $r_a r_b \neq 0$, $r_a + r_b = 2n$.

Since rang $\|a_{ik}\| = r_a$, there exist $n_1 = 2n - r_a$ linearly independent solutions $u_s(x, \lambda)$ $(s = 1, 2, \dots, n_1)$ of the equation

$$l(u) = \lambda u, \tag{2}$$

which satisfy the initial conditions

$$\sum_{k=1}^{2n} a_{rk} u_s^{[k-1]}(a, \lambda) = 0 \quad (r = 1, 2, \dots, 2n; \quad s = 1, 2, \dots, n_1). \tag{3}$$

Analogously, there exist $n_2 = 2n - r_b$ $(= 2n - n_1)$ linearly independent solutions $u_s(x, \lambda)$ $(s = n_1 + 1, \dots, 2n)$ of equation (2), which satisfy the initial conditions

$$\sum_{k=1}^{2n} b_{rk} u_s^{[k-1]}(b, \lambda) = 0 \quad (r = 1, 2, \dots, 2n; \quad s = n_1 + 1, \dots, 2n). \tag{4}$$

We shall call the Wronskian determinant $W(\lambda)$ of the solutions $u_s(x, \lambda)$ $(s = 1, 2, \dots, 2n)$, which satisfy the initial conditions (3) and (4), **normalized**. (We note that here and below it is assumed that the Wronskian determinant $W(\lambda)$ of the functions $u_s(x, \lambda)$ is formed from the quasi-derivatives of these functions. In that case $W(\lambda)$ does not depend on x.)

Theorem 1. *The zeros of the normalized Wronskian determinant are eigenvalues of the quasi-differential operator A.*

As for the converse assertion, it has been established only for $n = 1$. In the general case $(n > 1)$ the converse assertion is proved for non-real eigenvalues (Theorem 2).

2. Denote by $W_k(x, \lambda)$ the Wronskian determinant of the functions $u_s(x, \lambda)$ $(s = 1, \dots, k-1,\ k+1, \dots, 2n)$, and let

$$v_k(x, \lambda) = (-1)^k \frac{W_k(x, \lambda)}{W(\lambda)} \qquad (k = 1, 2, \dots, 2n)$$

be the adjoint system of solutions of equation (2). Then, under conditions (3) and (4), the Green function $\mathcal{G}(x, t, \lambda)$ of the operator A has the form

$$\mathcal{G}(x, t, \lambda) = \begin{cases} \displaystyle\sum_{k=1}^{n_1} u_k(x, \lambda) v_k(t, \lambda), & (t \geq x), \\ \displaystyle -\sum_{k=n_1+1}^{2n} u_k(x, \lambda) v_k(t, \lambda), & (t \leq x). \end{cases}$$

Now put

$$r_{mk}(\lambda) = \begin{cases} [\,u_m, u_k\,]_a, & (k = 1, 2, \dots, n_1), \\ [\,u_m, u_k\,]_b, & (k = n_1 + 1, \dots, 2n); \end{cases}$$

$$s_{mk}(\lambda) = \begin{cases} [\,v_k, v_m\,]_b, & (k = 1, 2, \dots, n_1), \\ [\,v_k, v_m\,]_a, & (k = n_1 + 1, \dots, 2n), \end{cases}$$

where $u_j = u_j(x, \lambda)$, $v_j = v_j(x, \lambda)$, and

$$[f, g] = \sum_{k=1}^{n} \{ f^{[k-1]} \overline{g}^{[2n-k]} - f^{[2n-k]} \overline{g}^{[k-1]} \}.$$

Then the auxiliary transformation $B_\lambda = iR_\lambda - iR_\lambda^* + 2 \operatorname{Im} \lambda \, R_\lambda^* R_\lambda$ of the operator A can be represented in the form

$$B_\lambda = \sum_{k,i=1}^{2n} (\,\cdot, \overline{v}_k) J_{ik} \overline{v}_i \qquad \left((f, g) = \int_a^b f \overline{g} \, dx \right),$$

where the matrix $J = \|J_{ki}\|$ is related to the matrices $R(\lambda) = \|r_{ik}(\lambda)\|$, $S(\lambda) = \|s_{ki}(\lambda)\|$ and to the Gram matrix G of the functions $v_k(x, \lambda)$ $(k = 1, 2, \dots, 2n)$ by the relation

$$J = \frac{1}{2 \operatorname{Im} \lambda} G^{-1} [S(\lambda) R(\lambda) - E]$$

(E is the identity matrix).

Using now the properties of the transformation B_λ (2) and the relation

$$\operatorname{Im}(Af, f) = \tfrac{1}{2} (B_\lambda \varphi, \varphi) \qquad (\varphi = (A - \lambda I) f),$$

we arrive at the following assertions:

I. Let, for some λ_0 $(\operatorname{Im} \lambda_0 \neq 0)$, the matrix J be Hermitian nonnegative (Hermitian nonpositive). Then the spectrum of the operator A is situated in the half-plane $\operatorname{Im} \lambda \geq 0$ $(\operatorname{Im} \lambda \leq 0)$.

II. The rank of the matrix $S(\lambda)R(\lambda) - E$ does not depend on λ. Moreover, if the operator A is self-adjoint, then for every λ $(\operatorname{Im} \lambda \neq 0)$

$$S(\lambda) R(\lambda) = E. \tag{5}$$

Conversely, if relation (5) holds for at least one nonreal λ, then the operator A is self-adjoint.

3. The preceding results make it possible to compute the characteristic matrix-function $\chi_A(\lambda)$ of the operator A, which for unbounded operators was introduced in (3). As a result we find that

$$\chi_A(\lambda) = W(\lambda) F^{-1}(\lambda), \tag{6}$$

where $W(\lambda)$ is the normalized Wronskian determinant, and $F(\lambda)$ is a certain matrix function whose determinant is a bounded function depending on the values of $[u_k, u_i]_x$ at the ends of the interval $[a, b]$. Using the results of [3] and relation (6), we arrive at the following assertion:

Theorem 2. *The nonreal eigenvalues of a regular quasi-differential operator A turn the normalized Wronskian determinant into zero.*

4. We now suppose that the operator A under consideration is simple [4] and dissipative [1]. This means that the operator A has no invariant subspaces on which $A^* = A$, and $\operatorname{Im}(Af, f) \geqslant 0$ for every $f \in D_A$. In this case the operator A has no real eigenvalues, its spectrum $\{\lambda_k\}_{k=1}^{\infty}$ is situated in the half-plane $\operatorname{Im}\lambda > 0$, and

$$\sum_{k=1}^{\infty} \operatorname{Im}\lambda_k = \infty.$$

(We note that a simple but not dissipative operator may have real eigenvalues.) Moreover,

$$\det(E - \tau G G^*) \leqslant \prod_{k=1}^{\infty} \left| \frac{\lambda_k - i}{\overline{\lambda}_k - i} \right|^2, \tag{7}$$

where G is the Gram matrix of the vectors $v_k(x, -i)$ $(k = 1, 2, \ldots, 2n)$, and τ is the matrix of transition from the system of vectors $\overline{v_k(x, -i)}$ $(k = 1, 2, \ldots, 2n)$ to the α-basis [3] of the operator A. The matrix τ is a rectangular matrix with $2n$ columns and r rows, where $r = \operatorname{rang}[S(-i)R(-i) - E]$.

Theorem 3. *The system of root subspaces of a simple dissipative quasi-differential operator A is complete in the space $L^2(a, b)$ if and only if the equality sign holds in relation (7).*

It is clear that the indicated criterion is applicable only to non-self-adjoint operators.

Uman
Pedagogical Institute

Received
22 VII 1963

CITED LITERATURE

1. I. M. Glazman, UMN, **13**, no. 3 (81) (1958).

2. A. V. Kuzhel, UMN, **16**, no. 3 (99) (1961).

3. A. V. Kuzhel, DAN, **125**, No. 1 (1959).

4. A. V. Kuzhel, DAN, **119**, No. 5 (1958).

Note: Figure translations are in progress. See original paper for figures.