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Abstract

Full Text

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ON THE SCATTERING OPERATOR IN THE SPACE OF SECOND QUANTIZATION

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In this paper the existence of the elastic part of the scattering operator is proved for energy operators of quantum-mechanical systems with a variable number of particles.

1. Systems with a variable number of particles are described by means of the apparatus of second quantization. Denote, as usual, by $a^*(\xi)$ and $a(\xi)$ the creation and annihilation operators,* where ξ is a parameter ranging over three-dimensional Euclidean space E_3. The operators $a^*(\xi)$ and $a(\xi)$ satisfy either the **Bose** commutation relations:

$$[a(x), a^*(y)] = \delta(x-y), \qquad [a(x), a(y)] = [a^*(x), a^*(y)] = 0, \tag{1}$$

or the **Fermi** relations:

$$\{a(x), a^*(y)\} = \delta(x-y), \qquad \{a(x), a(y)\} = \{a^*(x), a^*(y)\} = 0, \tag{2}$$

where $\delta(x-y)$ is the Dirac δ-function. As the space of states one considers a Hilbert space $\mathfrak{H}$, called the space of second quantization. It consists of vectors of the form

$$\Phi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int \varphi_n(x_1, \dots, x_n) a^*(x_1) \cdots a^*(x_n) d^n x \Phi_0 \quad \left(\sum_{n=0}^{\infty} \int |\varphi_n|^2 d^n x < \infty \right),$$

where Φ_0 is the vacuum vector, i.e. a solution of the equation $a(\xi)\Phi = 0$. The functions φ_n are symmetric in the coordinates $x_1, x_2, \dots, x_n$ in the Bose case and antisymmetric in the Fermi case. The scalar product in $\mathfrak{H}$ is given by the formula

$$(\Phi, \Psi) = \sum_{n=0}^{\infty} \int \varphi_n(x_1, \dots, x_n) \overline{\psi_n}(x_1, \dots, x_n) d^n x.$$

2. As energy operators of systems with a variable number of particles we shall consider operators defined in the space $\mathfrak{H}$ of the form**

$$H = H_0 + V$$

with natural domains of definition $D(H)$, where

$$H_0 = -\int a^*(\xi)\Delta_\xi a(\xi)\,d\xi,$$

$$V = \sum_{1\le k\le m\le M}(V_{mk} + V_{mk}^*), \qquad V_{mk} = \int a^*(\xi_1)\cdots a^*(\xi_m)\times$$

$$\times v_{mk}(\xi_1, \dots, \xi_m \mid \eta_1, \dots, \eta_k)a(\eta_1)\cdots a(\eta_k)d^m\xi\, d^k\eta. \tag{3}$$

The operators H_0 and V express, respectively, the kinetic energy of the system and the interaction energy. We emphasize that the operator V does not contain interactions of the form $V_{m0} + V_{m0}^*$. We shall assume the operators H to be self-adjoint.***

The following three theorems express sufficient conditions for the existence of the elastic part of the scattering operator.

* Strictly speaking, $a^*(\xi)$ and $a(\xi)$ are not operators but operator-valued generalized functions.

** The rules of action of the operator on a vector in the space $\mathfrak{H}$ are determined by relations (1) for a system of bosons and by relations (2) for a system of fermions.

*** In any case, the operator H is either symmetric or conjugate to a symmetric one. In the latter case, by H one means one of its self-adjoint restrictions.

Theorem 1. Suppose that the interaction operator V is such that, for all functions v_{mk} entering into its expression, the conditions

$$\int |v_{mk}(\xi_1, \dots, \xi_m \mid \eta_1, \dots, \eta_k)|^2 \prod_{i=1}^{m}\left(1 + |\xi_i|^{(2+\gamma)/m}\right) \times$$

$$\times \prod_{j=1}^{k}\left(1 + |\eta_j|^{(2+\gamma)/k}\right) d^m\xi\, d^k\eta \le A < \infty \tag{4}$$

are satisfied for some positive constants γ and A. Then the one-parameter family of operators

$$U_0(t) = \exp(iHt)\exp(-iH_0t)$$

has strong limits as $t \to \pm\infty$.

These limits

$$U_0^{\pm} = \lim_{t\to\pm\infty} U_0(t)$$

are called the wave operators of the elastic channel. The elastic part of the scattering operator is expressed through them by the formula

$$S_0 = (U_0^+)^* U_0^-.$$

Among interaction operators that do not satisfy the conditions of Theorem 1, of special interest are operators commuting with the total-momentum operator. They are expressed by formula (3) under the condition that the kernels of the operators V_{mk} can be represented in the form

$$v_{mk}(\xi_1, \dots, \xi_m \mid \eta_1, \dots, \eta_k) =$$

$$= \int w_{mk}(\xi_1 - u, \dots, \xi_m - u \mid \eta_1 - u, \dots, \eta_k - u)\, du, \tag{5}$$

where w_{mk} are arbitrary functions defined in the spaces $E_{3(m+k)}$. From (5) it follows that v_{mk} is a function of $m+k-1$ three-dimensional vectors:

$$v_{mk} = F_{mk}(\xi_2 - \xi_1, \dots, \xi_m - \xi_1 \mid \eta_1 - \xi_1, \dots, \eta_k - \xi_1). \tag{6}$$

In particular, v_{mk} may also be a generalized function:

$$v_{mk} = F_{mk}^{(s)}(\xi_2 - \xi_1, \dots, \xi_m - \xi_1 \mid \eta_{s+1} - \xi_1, \dots, \eta_k - \xi_1)\, \delta(\xi_1 - \eta_1) \dots$$

$$\dots \delta(\xi_s - \eta_s) + \dots, \tag{7}$$

where the dots denote symmetrization with respect to $\xi_1, \dots, \xi_m$ and $\eta_1, \dots, \eta_k$ separately. For interaction operators commuting with the total-momentum operator, the following analogues* of Theorem 1 are valid:

Theorem 2. Suppose

$$H = H_0 + \sum_{2\le k\le m\le M} (V_{mk} + V_{mk}^*),$$

where the kernels of the operators V_{mk} are given by formula (7) and satisfy the conditions:

$$F_{mk}^{(s)}(\mu_1, \dots, \mu_{m-1} \mid \nu_1, \dots, \nu_{k-s}) \in L_2(E_{3(m+k-s-1)}),$$

if $s \ge 2$, or

$$\int |F_{mk}^{(1)}(\mu_1, \dots, \mu_{m-1} \mid \nu_1, \dots, \nu_{k-1})|^2 \prod_{i=1}^{m-1} \left(1 + |\mu_i|^{(2+\gamma)/(m-1)}\right) \times$$

$$\times \prod_{j=1}^{k-1} \left(1 + |\nu_j|^{(2+\gamma)/(k-1)}\right) d^{m-1}\mu\, d^{k-1}\nu \le A < \infty$$

for some $\gamma > 0$, if $s = 1$. Then the wave operators $U_0^{\pm}$ exist.

Theorem 3. Suppose

$$H = H_0 + \sum_{2 \le k \le m \le M} (V_{mk} + V_{mk}^*),$$

where the kernels of the operators V_{mk} are given by formula (6) and satisfy the conditions

$$\int |F_{mk}(\mu_1, \dots, \mu_{m-1} \mid \nu_1, \dots, \nu_k)|^2 \prod_{i=1}^{m-1} (1 + |\mu_i|^{3+\gamma}) \prod_{j=1}^{k} (1 + |\nu_j|^{3+\gamma}) \times$$

$$\times d^{m-1}\mu\, d^k\nu \le A < \infty$$

for some $\gamma > 0$. Then the wave operators $U_0^{\pm}$ exist.

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[*] We emphasize that from the interaction operator (3) in Theorems 2 and 3 the term $V_{m1} + V_{m1}^*$ is excluded.

All three theorems are proved analogously. We shall dwell on the first.

3. Proof of Theorem 1. General remarks. The existence of the wave operators, by analogy with ordinary quantum mechanics (^{1}), is established on the basis of the finiteness of the lengths of the curves described in the Hilbert space $\mathfrak{h}$ by the vectors $U_0(t)\Phi$ as the argument t varies from 0 to $\pm\infty$. It is sufficient here that the set of vectors Φ not coincide with $\mathfrak{h}$, but form some everywhere dense set $\mathfrak{M}$ contained in $D(H)$. Thus the process of proof reduces to the choice of the sets $\mathfrak{M}$ in the Bose and Fermi cases and to establishing the integrability of the functions $\|V \exp(-iH_0 t)\Phi\|$ of the variable t for all $\Phi \in \mathfrak{M}$.

The Bose case. Consider vectors of the form

$$\Phi_{n,\lambda} = \frac{1}{\sqrt{n!}} \int \varphi(x_1, \lambda) \cdots \varphi(x_n, \lambda) a^*(x_1) \cdots a^*(x_n)\, d^n x\, \Phi_0,$$

where

$$\varphi(x, \lambda) = (2\pi)^{-3/4} \exp\{-\tfrac{1}{4}|x-\lambda|^2\}. \tag{8}$$

As the set $\mathfrak{M}$ one may take the linear span of the vectors $\Phi_{n,\lambda}$, where $n = 0, 1, \dots$, and the parameter λ runs through the space E_3.

The formulas below contain the function $\psi(x, \lambda, t)$, the solution of the Cauchy problem for the equation $i\dfrac{\partial\psi}{\partial t} = -\Delta_x \psi$ with initial condition (8). This solution is expressed by the formula

$$\psi(x, \lambda, t) = (2\pi)^{-3/4}(1+it)^{-3/2} \exp\left\{-\tfrac{1}{4}|x-\lambda|^2(1+it)^{-1}\right\} \tag{9}$$

and admits the estimate

$$|\psi(x, \lambda, t)| \le B|1+it|^{-(1+\delta)/k}|x-\lambda|^{-3/2+(1+\delta)/k},$$

where B is some positive constant, and $\delta \in (0, \frac{1}{2})$. On the basis of this estimate we obtain

$$\begin{aligned}
\|V_{mk}\exp(-iH_0 t)\Phi_{n,\lambda}\|^2 \le k! \int & v_{mk}(\xi_1, \dots, \xi_m \mid \eta_1, \dots, \eta_k) \\
& \times \bar{v}_{mk}(\xi_1, \dots, \xi_m \mid q_1, \dots, q_k) \prod_{1 \le j \le k} \psi(\eta_j, \lambda, t)\bar{\psi}(q_j, \lambda, t) \\
& \times \prod_{1 \le s \le n-k} |\psi(x_s, \lambda, t)|^2 \, d^m\xi \, d^k\eta \, d^{kq} \, d^{n-k}x \\
\le \frac{k! \, B^k}{|1+it|^{2+2\delta}} \int & v_{mk}(\xi_1, \dots, \xi_m \mid \eta_1, \dots, \eta_k)\bar{v}_{mk}(\xi_1, \dots, \xi_m \mid q_1, \dots, q_k) \\
& \times \prod_{1 \le j \le k} \left(|\eta_j - \lambda| \, |q_j - \lambda|\right)^{-3/2+(1+\delta)/k} \, d^m\xi \, d^k\eta \, d^{kq}.
\end{aligned}$$

To the last integral we apply the Cauchy—Bunyakovsky inequality, after first multiplying and dividing the integrand by

$$\prod_{1 \le j \le k} (1 + |\eta_j|^{(2+\gamma)/k})^{1/2}(1 + |q_j|^{(2+\gamma)/k})^{1/2}.$$

We now choose δ_0 from the interval $(0, \min(1/2, \gamma/2))$. Then

$$\int_{E_3} |\eta - \lambda|^{-3+(2+2\delta_0)/k}(1 + |\eta|^{(2+\gamma)/k})^{-1} \, d\eta \le c(\gamma, \delta_0) < \infty,$$

and conditions (4) lead to the inequality

$$\|V_{mk}\exp(-iH_0t)\Phi_{n,\lambda}\|^2 \le |1+it|^{-2-2\delta_0}\, k!\, B^k A\,[C(\gamma,\delta_0)]^k. \tag{10}$$

The integrability in t of the function $\|V\exp(-iH_0t)\Phi_{n,\lambda}\|$ is a consequence of inequality (10) and the inequality

$$\|V\Psi\|^2 \le \sum_{1\le k\le m\le M} \left(\|V_{mk}\Psi\|^2 + \|V^*_{mk}\Psi\|^2\right).$$

The passage from $\Phi_{k,\lambda}$ to an arbitrary vector $\Phi \in \mathfrak{M}$ presents no difficulty. Theorem 1 for the Bose case is proved.

Fermi case. Consider vectors of the form

$$\Phi_n(\lambda_1,\dots,\lambda_n) = \frac{1}{\sqrt{n!}}\int \varphi_n(x_1,\dots,x_n;\lambda_1,\dots,\lambda_n)\, a^*(x_1)\dots a^*(x_k)\, d^n x\, \Phi_0, \tag{11}$$

where $\varphi_n(x_1,\dots,x_n;\lambda_1,\dots,\lambda_n)$ is the determinant of the matrix whose elements φ_{ik} are the functions $\varphi(x_i,\lambda_k)$, defined by equality (8). The set $\mathfrak{M}$ is defined as the linear span of the vectors (11), where $n = 0,1,\dots$, and $\lambda_1,\lambda_2,\dots,\lambda_n$ is a system of independent parameters—vectors of the space E_3. It is dense in $\mathfrak{H}$ and is contained in $D(H)$.

Application of the operator $V_{mk}\exp(-iH_0t)$ to the vector (11) gives

$$\begin{aligned} V_{mk}\exp(-iH_0t)\Phi_n(\lambda_1,\dots,\lambda_n) &= \\ &= c_{mk}(n)\int a^*(\xi_1)\dots a^*(\xi_m)\, v_{mk}(\xi_1,\dots,\xi_m \mid \eta_1,\dots,\eta_k) \\ &\quad\times \psi_n(\eta_1,\dots,\eta_k,x_{k+1},\dots,x_n;\lambda_1,\dots,\lambda_n,t)\, a^*(x_{k+1})\dots a^*(x_n)\, d^m\xi\, d^{n-k}x\, d^k\eta, \end{aligned} \tag{12}$$

where

$$c_{mk}(n) = \begin{cases} \dfrac{1}{\sqrt{n!}}\, n(n-1)\dots(n-k+1), & \text{for } n \ge k, \\ 0, & \text{for } n < k, \end{cases}$$

and the function ψ_n is the determinant of the functions $\psi(x_i,\lambda_k,t)$, defined by equality (9). The norm of the vector corresponding to an individual term of the determinant ψ_n in expression (12) is estimated as in the Bose case. The number of such terms in (12) is finite. Thus, in the Fermi case as well, the integrability of the functions $\|V\exp(-iH_0t)\Phi\|$ is established for all $\Phi \in \mathfrak{M}$. This completes the proof of Theorem 1.

4. The obtained operators $U_0^{\pm}$ are isometric and, consequently, can be represented in the normal form (2). Moreover, they are transformation operators, i.e. $U_0^{\pm} H_0 = H U_0^{\pm}$.

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Note: Figure translations are in progress. See original paper for figures.

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