



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

I. A. KIYKO

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.02496>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1964, Vol. 157, No. 3

THEORY OF ELASTICITY

I. A. KIYKO

**A VARIATIONAL PRINCIPLE IN PROBLEMS
OF THE FLOW OF A THIN LAYER OF PLAS-
TIC MATERIAL**

(Presented by Academician A. Yu. Ishlinskii on 25 II 1964)

1. A layer of plastic material occupies in the plane xy an arbitrary region S , whose boundary Γ is a piecewise smooth curve $y_0 = \varphi(x_0)$, and is enclosed between the surfaces $z_1 = f_1(x, y)$ and $z_2 = f_2(x, y)$, bounding two elastic bodies which, approaching one another, cause the layer to spread; the functions f_1 and f_2 are continuous together with the derivatives of the necessary orders, and their first derivatives with respect to each of the coordinates are of order $h_0(x, y)/L \ll 1$, where L is a characteristic dimension of the region S , $h_0(x, y) = f_1(x, y) - f_2(x, y)$. The pressure in the layer is determined by the equation ⁽¹⁾

$$\left(\frac{dp}{dx}\right)^2 + \left(\frac{dp}{dy}\right)^2 = \frac{4\tau_S^2}{h^2} \quad (1)$$

and by the boundary condition

$$\text{on } \Gamma \quad p = \sigma_S, \quad (2)$$

where $\sigma_S = \sqrt{3}\tau_S$ is the yield strength of the layer material, and $h(x, y)$ is its thickness; the normal displacements of boundary points of the elastic bodies may be written in terms of the corresponding Green' s functions:

$$w_1 = \iint_S K_1(x, y; \xi, \eta) p(\xi, \eta) d\xi d\eta,$$

$$w_2 = \iint_S K_2(x, y; \xi, \eta) p(\xi, \eta) d\xi d\eta, \quad (3)$$

therefore, for the thickness of the layer we obtain

$$h(x, y) = h_0(x, y) + w_1(x, y) + w_2(x, y); \quad (4)$$

denoting $2\tau_S u = p$, $w = w_1 + w_2$, $K = (K_1 + K_2) 2\tau_S$, and adding term by term (3), we arrive at the system:

$$(\text{grad } u)^2 = \frac{1}{(h_0 + w)^2},$$

$$w = \iint_S K(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta, \quad (5)$$

$$\text{on } \Gamma \quad u = \sqrt{3}/2.$$

Two formulations of the problem are possible:

- a) the surfaces f_1 and f_2 are given, and it is required to determine the pressure distribution in the layer and the displacements w_1, w_2 ; the problem reduces to the simultaneous solution of system (5);
- b) determine the surfaces f_1 and f_2 such that at the end of the flow process a layer of prescribed thickness is obtained; in this case we arrive at the Cauchy problem for the first of equations (5), in which $h_0 + w$ is a known function.

In formulation a), one-dimensional and axisymmetric problems have been considered (^{2,3}); in formulation b), by the method of analogy with a sand heap (^{4,5}), or by reduction to a characteristic system, cases may be considered in which

thickness depends on one coordinate (⁶) or is constant; below a method is given for solving system (5) in the general case.

2. Let us consider the functional

$$U = \int_{\gamma} \frac{dx}{h(x, y)} \quad (6)$$

along a curve γ joining a given point of the region with the point (x_0, y_0) of the boundary Γ , which we shall define below; the extremum condition for this functional, written in Euler form, gives a field of extremals determined by the differential equation

$$y'' + (1 + y'^2) \left(\frac{\partial \ln h}{\partial y} - \frac{\partial \ln h}{\partial x} y' \right) = 0, \quad (7)$$

depending on two parameters and having the boundary Γ as a transversal; the transversality condition in the present case reduces to the orthogonality condition ⁽⁷⁾

$$y'(x_0)\varphi'(x_0) = -1 \quad (8)$$

serving to determine the point (x_0, y_0) .

The extremum condition for (6), written in Hamilton form, leads, as is not difficult to show ⁽⁷⁾, to the equation

$$(\text{grad } U)^2 = \frac{1}{h^2(x, y)}, \quad (9)$$

which shows that the flow problem and the problem of finding the extremum of the functional (6) are equivalent, and that the solution of the former can be written in the form

$$u = \frac{V\sqrt{3}}{2} + \int_{\gamma} \frac{ds}{h(x, y)}, \quad (10)$$

where the integral is taken along the extremal ⁽⁷⁾; hence it is seen that the field of extremals gives a family of streamlines for the flow problem.

In formulation b) the problem is now solved as follows: from ⁽⁷⁾ one finds the family of extremals, from (10) the pressure in the layer, and from (3) the displacements, after which we obtain the required surfaces:

$$f_1 = \zeta_1 - w_1; \quad f_2 = \zeta_2 + w_2,$$

where ζ_1 and ζ_2 are the prescribed surfaces determining the thickness of the layer at the end of the process. For the solution in formulation a), a method of successive approximations may be proposed, though without specifying convergence: set $w^0 = 0$, i.e. $h^0 = h_0$; from ⁽⁷⁾, accordingly, find y^0 , and from (10), u^0 ; substituting this into the second of equations (5), determine w^1 , and then $h^{(1)} = h_0 + w^{(1)}$; this is substituted into ⁽⁷⁾, and so on.

Let us note that the general solution of ⁽⁷⁾ can be found only in special cases; however, the use of direct methods—for example, those of Ritz or Galerkin—makes it possible to investigate the problem in sufficiently general cases.

Let us also point out the analogy of the stated flow problem with geometrical optics: the problem of the extremum of the functional (6) is Fermat's principle; the extremals are the trajectories of rays in a medium with a given refractive index, and the transversals are wave surfaces, or eikonals.

3. As an example, suppose that the thickness depends on one coordinate; in this case (7) is solved in quadratures

$$y + b = a \int \frac{h(x) dx}{[1 - a^2 h^2(x)]^{1/2}} \quad \text{or} \quad x + a_1 = b_1 \int \frac{h(y) dy}{[1 - b_1^2 h^2(y)]^{1/2}} \quad (11)$$

and yields a result known in geometrical optics as Bouguer' s theorem ⁽⁸⁾: the angle α between the direction in which the thickness changes and the tangent to the extremal (streamline) is such that the ratio $h/\sin \alpha$ remains constant along the extremal (streamline); in those cases when the integrals (11) cannot be taken, this theorem makes it possible to construct the system of streamlines graphically.

ically. If $h = h_0(1 + \lambda x)$, then the streamlines will be circles whose centers lie on a straight line parallel to the y -axis and intersecting the x -axis at the point $-1/\lambda$; for a circular region $x_0^2 + y_0^2 = R^2$ we find that all streamlines gather at the point

$$x_1 = (\sqrt{1 - \lambda^2 R^2} - 1) / \lambda.$$

Received
10 II 1964

REFERENCES

1. A. A. Ilyushin, *Prikl. matem. i mekh.*, No. 3 (1954).
2. I. A. Kiiko, *Izv. AN SSSR, Mekh. i mashinostr.*, No. 6 (1961).
3. I. A. Kiiko, *Inzh. zhurn.*, 3, issue 1 (1963).
4. A. A. Ilyushin, *Prikl. matem. i mekh.*, No. 6 (1955).
5. P. M. Ogibalov, I. A. Kiiko, *Inzh. zhurn.*, No. 3 (1961).
6. Yu. S. Arutyunov, A. L. Gonor, *Izv. AN SSSR, Mekh. i mashinostr.*, No. 1 (1963).
7. M. A. Lavrent' ev, L. A. Lyusternik, *Course of Variational Calculus*, Moscow—Leningrad, 1950.
8. M. Herzberger, *Modern Geometrical Optics*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.