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Abstract

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SINGULAR ELLIPTIC EQUATIONS AND SYSTEMS OF VARIABLE ORDER

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1. Singular elliptic operators (s.e.o.) of variable order on a closed manifold. Let an equation be given on a closed manifold M^n

$$L_{\alpha(x)}u \equiv K_{\alpha(x)}u + Tu = f(x), \qquad x \in M^n, \tag{1}$$

where the operator $K_{\alpha(x)}$ in a neighborhood $V_j \subset M^n$ has the form

$$K_{\alpha(x)}\varphi_j u \equiv \int_{R^n} K_{\alpha(x)}(x, x-y)\varphi_j(y)u(y)\,dy + T_1\varphi_j u, \qquad x \in V_j; \tag{2}$$

$\operatorname{supp}\varphi_j \subset V_j$, $\varphi_j \in C^\infty$. It is assumed that the symbol of the operator $K_{\alpha(x)}$, i.e. the Fourier transform $K_{\alpha(x)}(x,z)$ with respect to z: $F_z K_{\alpha(x)}(x,z) = \widetilde{K}_{\alpha(x)}(x,\xi)$, is a homogeneous function of ξ of order $\alpha(x)$, so that $\widetilde{K}_{\alpha(x)}(x,t\xi) = t^{\alpha(x)}\widetilde{K}_{\alpha(x)}(x,\xi)$. For simplicity let $\widetilde{K}_{\alpha(x)}(x,\xi)$ be infinitely differentiable with respect to x and ξ ($\xi \neq 0$). The operators T and T_1 in (1) and (2) are subordinate. We shall say that $K_{\alpha(x)}$ is a **singular elliptic operator** (s.e.o.) if $\widetilde{K}_{\alpha(x)}(x,\xi) \neq 0$, $x \in M^n$, $\xi \neq 0$.

2. Spaces of functions of variable order of smoothness. We introduce the space $H^{\beta(x)}(M^n)$ with norm $\|u\|_{\beta(x)} = \|S_{\beta(x)}u\|_0 + \|u\|_{\beta_0}$, where $S_{\beta(x)}$ is an s.e.o. with symbol $(1+|\xi|)^{\beta(x)}$, $\beta(x) \in C^\infty(M^n)$, $\beta_0 = \min\beta(x) - 1$, and $\|u\|_l$ is the usual norm of order l.

Theorem 1. 1) If $\beta_1(x) < \beta_2(x)$, then $H^{\beta_2(x)} \subset H^{\beta_1(x)}$, and the embedding operator is completely continuous.

2) Let $\{V_j\}$ be a covering of M^n, $\{\varphi_j\}$ the corresponding partition of unity, and let $(\beta) = \{\beta_j\}$, $(\gamma) = \{\gamma_j\}$ be such sets of numbers that $\beta_j < \beta(x) < \gamma_j$ for $x \in \overline{V}_j$.

Then $H^{(\gamma)} \subset H^{\beta(x)} \subset H^{(\beta)}$, and the embedding operators are completely continuous; the norm in $H^{(\delta)}$ is given by the formula $\|u\|_{(\delta)} = \sum \|\varphi_j u\|_{\delta_j}$.

3. Normal solvability of equation (1) in $H^{\beta(x)}$. Theorem 2. The operator L is a Φ-operator (1) acting from the space $H^{\beta(x)}$ into $H^{\beta(x)-\alpha(x)}$, if $K_{\alpha(x)}$ is an s.e.o., and T is a completely continuous operator from $H^{\beta(x)}$ into $H^{\beta(x)-\alpha(x)}$. The estimate holds

$$\|u\|_{\beta(x)} \leq C\left(\|f\|_{\beta(x)-\alpha(x)} + \|u\|_{\beta_0}\right). \tag{3}$$

Here and below, as a subordinate operator T it suffices to take an operator of the form $\sum K_{\gamma_i(x)} + T_N$, where $\widetilde{K}_{\gamma_i(x)}(x,\xi) = |\xi|^{\gamma_i(x)} \ln^{\delta_i} |\xi|\, \widetilde{K}_{0,i}(x,\xi)$; $\gamma_i(x) < \alpha(x)$; $\widetilde{K}_{0,i}(x,t\xi) = \widetilde{K}_{0,i}(x,\xi)$; T_N is a smoothing operator of order-

order N, i.e., if $u \in H^{\alpha_0}$, then $T_N u \in H^{N+\alpha_0}$. N is fixed, sufficiently large, and in each concrete case its value can be specified.

4. Boundary-value problems for s.e.o. in a bounded domain. We now consider an equation of the form (1) in a bounded domain $G \subset R^n$ with smooth boundary Γ. As in (^{2}), we factor the kernel $\widetilde{K}_{\alpha(x)}$ on Γ:

$$\widetilde{K}_{\alpha(x)}(x,\xi) = \widetilde{K}^{+}_{\chi(x)}(x,\xi)/\widetilde{K}^{-}_{\chi(x)-\alpha(x)}(x,\xi),$$

$\xi = (\xi', \xi_n)$, ξ_n corresponds to the normal to Γ at the point $x \in \Gamma$; $\widetilde{K}^{+}_{\chi(x)}(x,\xi) \neq 0$, $\operatorname{Im}\xi_n \geqslant 0$, $\widetilde{K}^{-}_{\chi(x)-\alpha(x)}(x,\xi) \neq 0$, $\operatorname{Im}\xi_n \leqslant 0$, $\xi \neq 0$. In (^{2}) general boundary-value problems were considered for the case of constant and integral χ and for $\widetilde{K}^{+}_{\chi}$ satisfying condition c) or c$'$). Here the general case of nonintegral and variable $\chi(x)$ is considered. We extend $\widetilde{K}^{+}_{\chi(x)}$ to the whole space R^n, preserving the ellipticity condition: $\widetilde{K}^{+}_{\chi}(x,\xi) \neq 0$, $x \in R^n$, $\xi \neq 0$. It is proved that there exists the following representation of the operator $L_{\alpha(x)}$ in all of R^n:

$$L_{\alpha(x)} = L^{-} \cdot L^{+} + T_N, \tag{4}$$

where L^{+} is a sum of operators with symbols $\widetilde{K}^{+}_{\chi(x)}(x,\xi)$, $\widetilde{K}^{+}_{\gamma_i(x)}(x,\xi)$, $\gamma_i(x) < \chi(x)$, and all of them, for $x \in \Gamma$, are analytic for $\operatorname{Im}\xi_n > 0$. Similarly, L^{-} is a sum of operators with symbols $[\widetilde{K}^{-}_{\chi(x)-\alpha(x)}(x,\xi)]^{-1}$, $\widetilde{K}^{-}_{\delta_j(x)}(x,\xi)$, $\delta_j(x) < \alpha(x) - \chi(x)$, analytic in ξ_n for $x \in \Gamma$, $\operatorname{Im}\xi_n < 0$. By $u(x)$ we shall denote, generally speaking, generalized functions defined in R^n and equal to zero in $R^n \setminus (G \cup \Gamma)$. Let $v(x) = D^{+}L^{+}u$, where D^{+} is an s.e.o. of order of homogeneity $-M$ with symbol $\xi_{+}^{-M} = (\xi_n + i|\xi'|)^{-M}$ on Γ, M an integer, $M \geqslant 0$. By $H^{l(x)}_{L^{+},M}(G)$ we denote the space of functions $u(x)$ such that $v(x) \in L^2_{\mathrm{loc}}(R^n)$ with finite norm

$$\|u\|_{l(x)} = \|v(x)\|_{l(x)-\chi(x)+M} + \|u\|_{-N}, \tag{5}$$

where $\|v\|_{\gamma(x)} = \inf \|Lv\|_{\gamma(x)}$, Lv is an extension of v from G to R^n. The latter means that $v(x) \in H^{\gamma(x)}(G)$. We note that the functions $u(x) \in H^{l(x)}_{L^+,M}$ have smoothness of order $l(x)$ inside G, and near Γ, $u(x) = O(r^{\chi(x)-M})$ for nonintegral $\chi(x)$; r is the distance of x from Γ. If L^+ satisfies condition c) from (^{2}) and, hence, χ is integral, then $u(x)$ may turn into a δ-function and its derivatives on Γ when $M > \chi$. Therefore, usually in the case of boundary-value problems for differential equations one takes $M = \chi$, and then $H^l_{L^+,M} = H^l$ $(l(x) \equiv l)$. In the general case there is no necessity for this, and M indicates the necessary number of boundary conditions. We note that when $M < \chi(x)$, part of the boundary conditions is contained in the fact that $u \in H^{l(x)}_{L^+,M}$.

On Γ we prescribe M boundary conditions of the form

$$B_j u\big|_\Gamma = F_j(x'), \qquad x' \in \Gamma \quad (j = 1, \dots, M), \tag{6}$$

where B_j are operators of the form $L_{\alpha(x)}$, satisfying the conditions:

1) $B_j \cdot R^+ = V_j + T_{j,N}$, where $L^+ \cdot R^+ = I + T_N$, and for V_j condition c) from (^{2}) holds.

2)

$$\det \left\| \int_{\Gamma_0} B_j^{(0)}(x,\xi)\, [\widetilde{K}^+_{\chi(x)}(x,\xi)]^{-1}\, \xi_n^{k-1}\, d\xi_n \right\| \neq 0, \qquad x \in \Gamma, \quad \xi' \neq 0, \quad j,k = 1, \dots, M,$$

where $B_j^{(0)}(x,\xi)$ is the principal part of $\widetilde{B}_j(x,\xi)$, and Γ_0 is the same contour as in (^{2}). We note that in a number of cases condition 1) can be dropped.

Theorem 3. *If $L_{\alpha(x)}$ is an s.e.o. and B_j $(j = 1, \dots, M)$ satisfy conditions 1) and 2), then the operator corresponding to problem (1), (6) is a Φ-operator in the corresponding spaces, and the estimate holds*

$$\|u\|_{l(x)} \leqslant C \left(\|f\|_{l(x)-\alpha(x)} + \sum_{j=1}^{M} \|F_j\|_{l(x)-m_j(x)-1/2} + \|u\|_{l(x)-1} \right), \tag{7}$$

where $f \in H^{l(x)-\alpha(x)}(G)$, $F_j \in H^{l(x)-m_j(x)-1/2}(\Gamma)$, $m_j(x) = \operatorname{ord} \widetilde{B}_j$, $l(x) > \max(m_j(x) + 1/2, \alpha(x), \varkappa(x) - M)$.

This theorem, in the case $\widetilde{K}^+$ satisfying condition c) and $M = \varkappa$, coincides with Theorem 3 in (^{2}).

5. Problems with additional potentials.

In the case of an integer negative M, in the norm of the space $H^{\chi(x)}_{l+,M}$, instead of the boundary conditions (6), one should add to equation (1) $|M|$ terms of potential type (see (2)). Thus, the equation considered is

$$L_{\alpha(x)}\left(u(x) + \sum_{k=1}^{|M|} G_k g_k(x')\right) = f(x), \qquad x \in G, \quad x' \in \Gamma, \tag{8}$$

where G_k are operators of the type $L_{\alpha(x)}$ (see (2)), satisfying the following conditions:

1′) $L^+ G_k = W_k + T_{k,N}$, where W_k satisfies condition c) from (2).

2′) Since $\widetilde{W}_k^{(0)}$ satisfies condition c), on Γ $\xi_+^M \Pi^+ \widetilde{W}_k^{(0)}(x,\xi) = P_k(x,\xi',\xi_n) + R_k^+(x,\xi)$, where $\widetilde{W}_k^{(0)}$ is the principal part of $\widetilde{W}_k$, $\widetilde{W}_k^{(0)} = \Pi^+\widetilde{W}_k^{(0)} + \widetilde{W}_k^-$, $\Pi^+\widetilde{W}_k^{(0)}$ is analytic for $\operatorname{Im}\xi_n > 0$ and decreases as $\xi_n \to \infty$, $P_k(x,\xi',\xi_n)$ are polynomials in ξ_n of degree not higher than $|M|-1$, $|R_k^+(x,\xi)| \leq C(x,\xi')/(|\xi_n| + |\xi'|)$.

Condition 2′) on G_k consists in the fact that the polynomials P_k $(k = 1, \dots, |M|)$ are linearly independent.

Theorem 4. *The operator corresponding to equation (8), where $L_{\alpha(x)}$ is a properly elliptic operator and G_k satisfy conditions 1′), 2′), is a Φ-operator in the corresponding spaces; moreover, the estimate*

$$\|u\|_{l(x)} + \sum_{k=1}^{|M|} \|g_k\|_{l(x)+\alpha_k(x)+1/2} \leq C\left(\|f\|_{l(x)-\alpha(x)} + \|u'\|_{l(x)-1} + \sum_{k=1}^{|M|} \|g_k\|_{\delta_k(x)}\right) \tag{9}$$

$$\alpha_k(x) = \operatorname{ord} G_k, \qquad \delta_k(x) = l(x) + \alpha_k(x) - 1/2.$$

Let us note that the solution $u(x)$ of equation (8) has smoothness $\varkappa(x) + |M|$ up to the boundary Γ. This is achieved by isolating terms of the form of potentials $G_k g_k(x')$, which contain the principal singularities of the solution of the equation $L_{\alpha(x)} v = f$ in G.

Remark. a) We note that the potentials in (8) may be taken in the same form as in (2), i.e., the left-hand side of (8) may be replaced by the following: $L_{\alpha(x)} u + \sum \widehat{G_k} g_k$. b) The number of potentials may be increased to the number $|M| + s$, adding at the same time additionally s boundary conditions of the form (6).

Example. We restrict ourselves to the simplest example. Let, for simplicity, in (1) $T = 0$ and let $K_{\alpha(x)}$ be the operator with symbol $|\xi|^\alpha$ (α constant), G a bounded domain. Then for $\alpha < 0$, K_α is an integral operator. For $\alpha > 0$, K_α is an integro-differential operator. $\widetilde{L}^+ = \xi_+^{\alpha/2}$, so that $\varkappa = \alpha/2$. Let $\alpha > 0$. Take $M = [\alpha/2]$ and denote $\gamma = \alpha/2 - M$. Then as boundary operators B_j one may take operators with symbols $\widetilde{B}_j = \xi_+^{\gamma+j-1}$ $(j = 1, \dots, M)$, and all the conditions of Theorem 3 will be fulfilled. If $\alpha < 0$, we take, for example, M such that $\alpha/2 = -M + \gamma$, $0 \le \gamma < 1$. As G_k in (8) it is sufficient to take kernels with symbols $\xi_+^{M-1-k-\gamma}$ $(k = 1, \dots, |M|)$, in order that all the conditions of Theorem 4 be fulfilled.

6. Systems of properly elliptic operators. Consider a system of equations of the form (1) in the domain G with symbol $\widetilde{K}_{\alpha(x)}(x, \xi)$, which is a matrix of order $r \times r$. The ellipticity condition now consists in the fact that $\det \widetilde{K}_{\alpha(x)}(x, \xi) \neq 0$, $\xi \neq 0$, $x \in G \cup \Gamma$, while the elements $\widetilde{K}_{\alpha(x)}(x, \xi)$ are homogeneous functions in ξ of order $\alpha(x)$. We shall call the system properly elliptic at points $x \in \Gamma$ if $\widetilde{K}_{\alpha(x)}(x, \xi)$ admits a factorization of the form (see (^{3}))

$$\widetilde{K}_{\alpha(x)}(x, \xi) = \widetilde{K}_-^{-1}(x, \xi) S^{-1} S_+ \widetilde{K}_+(x, \xi), \tag{10}$$

where $\widetilde{K}_+$, $\widetilde{K}_-$ are homogeneous matrices of order zero with determinant not vanishing for $\operatorname{Im} \xi_n \geqslant 0$, $\operatorname{Im} \xi_n \leqslant 0$ $(\xi \neq 0)$, respectively; $S_+ = \|\delta_{jk} \xi_+^{\varkappa_j(x)}\|$, $S_- = \|\delta_{jk} \xi_+^{\varkappa_j(x) - \alpha(x)}\|$, $\widetilde{K}_+$, $\widetilde{K}_-$ depend smoothly on x and ξ $(\xi \neq 0)$. Let L^+, L^-, R^+ be operator matrices defined in the same way as in the scalar case, i.e. $L_{\alpha(x)} = L^- L^+ + T_N$, $x \in R^n$, $L^+ R^+ = I + T_N$. Let, further, $M = (M_1, \dots, M_r)$ be an arbitrary integer vector, and, for definiteness, $M_i > 0$, $1 \leqslant i \leqslant k$, $M_j \leqslant 0$, $k + 1 \leqslant j \leqslant r$. Denote

$$M_+ = \sum_{i=1}^{k} M_i, \qquad M_- = \sum_{j=k+1}^{r} |M_j|.$$

Introduce the space $H^l_{L^+, M}(G)$ with norm

$$|||u|||_{l(x)} = \sum_{k=1}^{r} \|v_k\|_{\gamma_k(x)},$$

where $v = (v_1, \dots, v_r)$,

$$v = D^+ L^+ u, \qquad \widetilde{D}^+ = \|\xi_+^{-M_i} \delta_{ik}\|, \qquad \gamma_k(x) = l(x) - \varkappa_k(x) + M_k;$$

$\| \ \|_{\gamma_k(x)}$ is the norm in $H^{\gamma_k(x)}(G)$ (see item 4).

The boundary-value problem for (1) in this case is posed as follows:

$$L_{\alpha(x)}(u + G_1 g) = f(x), \qquad x \in G, \tag{11}$$

$$B(u + G_1 g)\big|_\Gamma = F(x'), \qquad x' \in \Gamma, \tag{12}$$

where G_1 is a matrix of order $r \times M_-$, and the matrix B is of order $M_+ \times r$. We formulate conditions on B and on G_1:

1) $B \cdot R^+ = V + T_N$, where V satisfies condition c).

2) Let V_0 be the principal part of V. Denote by $P(x,\xi',\xi_n)$ the vector-polynomial $(P_1,\dots,P_r)$ such that $P_j \equiv 0$ for $j \geqslant k+1$, while for $1 \leqslant j \leqslant k$, P_j is an arbitrary polynomial in ξ_n of degree $M_j - 1$ with coefficients depending on ξ' and x. Let Z be a matrix of order $r \times M_+$ with entries equal to zero or to ξ_n^i, such that $P = ZC$, where C is an M_+-dimensional vector with components equal to the coefficients of the polynomials P_j. It is required that

$$\det \int_{\Gamma_0} \widetilde{V}_0(x,\xi) Z \, d\xi_n \neq 0, \qquad \xi' \neq 0, \quad x \in \Gamma.$$

1′) $G_1 = R^+ W + T_N$, where the matrix $W = \|W_{jk}\|$ satisfies condition c).

2′) Let $W_{jk}^{(0)}$ be the principal part of W_{jk}. We have:

$$\xi_+^{-M_j} \Pi^+ W_{jk}^{(0)}(x,\xi) = P_{jk}(x,\xi) + R_{jk}^+(x,\xi),$$

where

$$|R_{jk}^+(x,\xi)| \leqslant C(x,\xi')/(|\xi_n| + |\xi'|),$$

$P_{jk}(x,\xi)$ are polynomials in ξ_n, and $P_{jk} \equiv 0$ for $1 \leqslant j \leqslant k$, while $\operatorname{ord} P_{jk} \leqslant |M_j| - 1$ for $k+1 \leqslant j \leqslant r$. Condition 2′) on G_1 consists in the fact that the vector-polynomials

$$P_l = (P_{k+1,l},\dots,P_{r,l})$$

are linearly independent ($l = 1,\dots,M_-$).

Theorem 5. *If conditions 1), 2), 1′, 2′ are fulfilled, then problem (11), (12) for a system properly elliptic on Γ is normally solvable in the corresponding spaces, and the estimate*

$$|||u|||_{l(x)} + \sum_{k=1}^{M_-} \|g_k\|_{l(x)+\alpha_k(x)+1/2}$$

$$\leqslant C\left(\|f\|_{l(x)-\alpha(x)} + \sum_{k=1}^{M_+} \|F_k\|_{l(x)-m_{\tilde{k}}(x)-1/2} + |||u|||_{l(x)-1} + \sum_{k=1}^{M_-} \|g_k\|_{l(x)+\alpha_k(x)-1/2} \right),$$

where $B = \|B_{ik}\|$, $\operatorname{ord} \widetilde{B}_{ik} = m_i(x)$, $G_1 = \|G_{1,ik}\|$, $\operatorname{ord} \widetilde{G}_{1,ik} = \alpha_k(x)$, $u \in H^l_{L^+,M}$, $f \in H^{\,l(x)-\alpha(x)}(G)$, $F_k \in H^{\,l(x)-m_k(x)-1/2}(\Gamma)$, $g_k \in H^{\,l(x)+\alpha_k(x)+1/2}(\Gamma)$.

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