



Soviet-era science, translated into English

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1964

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Abstract

Full Text

MATHEMATICS

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INVARIANCE OF CERTAIN SPACES ASSOCIATED WITH THE OPERATOR $A - \lambda I$

(Presented by Academician V. I. Smirnov on 21 IX 1963)

Let A be a linear closed operator with domain of definition \mathfrak{D}_A and range \mathfrak{R}_A in a Banach space \mathfrak{B} . Consider the operator $A_\lambda = A - \lambda I$, where I is the identity operator and λ is a complex parameter. Denote by $\alpha(A_\lambda)$ the dimension of the null space \mathfrak{Z}_{A_λ} of the operator A_λ , and by $\beta(A_\lambda)$ the dimension of the quotient space $\mathfrak{B}/\mathfrak{R}_{A_\lambda}$. The set Φ of all those values λ for which \mathfrak{R}_{A_λ} is closed and at least one of the numbers $\alpha(A_\lambda)$, $\beta(A_\lambda)$ is finite is an open set. For any $\lambda \in \Phi$ put

$$\mathfrak{N}_\lambda = \bigcup_{n=1}^{\infty} \mathfrak{Z}_{A_\lambda^n}, \quad \mathfrak{M}_\lambda = \bigcup_{n=1}^{\infty} \mathfrak{R}_{A_\lambda^n}.$$

The elements of \mathfrak{N}_λ and \mathfrak{M}_λ are called, respectively, the **nil-elements** and the **elements of the core** of the operator A_λ . Obviously, the sets \mathfrak{N}_λ and \mathfrak{M}_λ are invariant with respect to A_λ , and moreover \mathfrak{M}_λ , being the intersection of closed sets, is closed. It can be proved that the operator A_λ maps \mathfrak{M}_λ onto itself, i.e. $A_\lambda(\mathfrak{D}_A \cap \mathfrak{M}_\lambda) = \mathfrak{M}_\lambda$ (for the case when $\alpha(A_\lambda) < \infty$, this was established in ⁽¹⁾).

Our task is to study the sets \mathfrak{N}_λ and \mathfrak{M}_λ as λ varies. This will be done by representing the elements of \mathfrak{M}_λ in the form of analytic functions of the parameter λ (see ⁽²⁾).

Let $A_{1\lambda}$ be the restriction of the operator A_λ to $\mathfrak{D}_{1\lambda} = \mathfrak{D}_A \cap \mathfrak{M}_\lambda$. It induces an operator $\tilde{A}_{1\lambda}$ in the quotient space $\mathfrak{M}_\lambda/\mathfrak{Z}_{A_{1\lambda}} \subset \mathfrak{B}/\mathfrak{Z}_{A_{1\lambda}}$ (here $\mathfrak{Z}_{A_{1\lambda}} = \mathfrak{Z}_{A_\lambda} \cap \mathfrak{M}_\lambda$). Let $N_{1\lambda} = \|\tilde{A}_{1\lambda}^{-1}\|$. Fix some value $\lambda_0 \in \Phi$ and some element $y_0 \in \mathfrak{M}_{\lambda_0}$. The equation $A_{\lambda_0} y = y_0$ has a solution $y_1 \in \mathfrak{M}_{\lambda_0}$, which may be chosen so that $\|y_1\| \leq (N_{1\lambda_0} + 1)\|y_0\|$. Similarly, the equation $A_{\lambda_0} y = y_1$ has a solution $y_2 \in \mathfrak{M}_{\lambda_0}$ such that $\|y_2\| \leq (N_{1\lambda_0} + 1)\|y_1\|$, and so on. Form the series

$$y_{1\lambda} = y_1 + (\lambda - \lambda_0)y_2 + (\lambda - \lambda_0)^2 y_3 + \dots$$

and the derived series

$$y_{1\lambda}^{(n)} = \frac{1}{n!} \frac{d^n}{d\lambda^n} y_{1\lambda}, \quad n = 1, 2, \dots,$$

all of which converge in the disk $|\lambda - \lambda_0| < 1/(N_{1\lambda_0} + 1)$. Direct verification shows that

$$A_\lambda^n y_{1\lambda}^{(n-1)} = y_0, \quad n = 1, 2, \dots$$

Consequently, $y_0 \in \mathfrak{M}_\lambda$. By the arbitrariness of the element y_0 of \mathfrak{M}_{λ_0} , we obtain the inclusion $\mathfrak{M}_{\lambda_0} \subset \mathfrak{M}_\lambda$, where $\lambda \in \Phi$ and $|\lambda - \lambda_0| < 1/(N_{1\lambda_0} + 1)$. Note that termwise application of the operator A_λ to convergent series is possible by virtue of the closedness of this operator.

Theorem 1. *The set Γ of those values λ in Φ for which $\mathfrak{Z}_{A_\lambda} \neq \mathfrak{Z}_{A_{\lambda_0}}$ is an isolated set.*

In fact, every zero of the operator A_λ , $\lambda \neq \lambda_0$, belongs to \mathfrak{M}_{λ_0} (for, if $Ax - \lambda x = 0$, then $x = (\lambda - \lambda_0)^{-n} A_{\lambda_0}^n x$ for arbitrary n), i.e. $\mathfrak{Z}_{A_\lambda} \in \mathfrak{M}_{\lambda_0}$. But, as was shown, $\mathfrak{M}_{\lambda_0} \subset \mathfrak{M}_\lambda$ in the circle $|\lambda - \lambda_0| < 1/(N_{\lambda_0} + 1)$; consequently, $\mathfrak{Z}_{A_\lambda} \in \mathfrak{M}_\lambda$. Thus, $\lambda \in \Gamma$ for all λ from this circle, except, possibly, $\lambda = \lambda_0$. The theorem is proved.

Consider the set $\mathfrak{N}_{\lambda_0} \cap \mathfrak{M}_{\lambda_0}$ and fix in it some nonzero element \bar{x} . Let n_0 be the smallest of the numbers n , $n = 1, 2, \dots$, for which $A_{\lambda_0}^{n_0} \bar{x} = 0$. Starting from the element \bar{x} , denoted henceforth by x_{n_0-1} , construct a sequence $x_{n_0-1}, x_{n_0}, x_{n_0+1}, \dots$ of elements from \mathfrak{M}_{λ_0} such that

$$A_{\lambda_0} x_{n+1} = x_n, \quad \|x_{n+1}\| \leq (N_{\lambda_0} + 1) \|x_n\|, \quad n = n_0 - 1, n_0, n_0 + 1, \dots$$

If $n_0 > 1$, then put $x_{n_0-2} = A_{\lambda_0} x_{n_0-1}$, $x_{n_0-3} = A_{\lambda_0} x_{n_0-2}, \dots, x_0 = A_{\lambda_0} x_1$. Obviously, $x_n \in \mathfrak{N}_{\lambda_0} \cap \mathfrak{M}_{\lambda_0}$, $n = 0, 1, 2, \dots$, and x_0 is a zero of the operator A_{λ_0} . Form the series

$$x_{0\lambda} = x_0 + (\lambda - \lambda_0)x_1 + (\lambda - \lambda_0)^2 x_2 + \dots$$

and the derived series

$$x_{0\lambda}^{(n)} = \frac{1}{n!} \frac{d^n}{d\lambda^n} x_{0\lambda} = \frac{1}{n!} \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} (\lambda - \lambda_0)^{k-n} x_k, \quad n = 1, 2, \dots$$

All these series converge in the circle $|\lambda - \lambda_0| < 1/(N_{\lambda_0} + 1)$. The relation between the sequences $\{x_n\}$ and $\{x_{0\lambda}^{(n)}\}$ can also be expressed by the series

$$x_n = \frac{1}{n!} \sum_{k=0}^{\infty} (-1)^k \frac{(n+k)!}{k!} (\lambda - \lambda_0)^k x_{0\lambda}^{(n+k)}, \quad n = 0, 1, 2, \dots,$$

convergent in the circle $|\lambda - \lambda_0| < 1/2(N_{\lambda_0} + 1)$. Hence it follows that all x_n (and, in particular, x_{n_0-1}) belong to $\overline{\mathfrak{N}}_\lambda$, since the sequence $\{x_{0\lambda}^{(n)}\}$ forms an infinite chain of zero-elements from \mathfrak{N}_λ :

$$A_\lambda x_{0\lambda} = 0, \quad A_\lambda x_{0\lambda}^{(n)} = x_{0\lambda}^{(n-1)}, \quad n = 1, 2, \dots$$

Thus, $x_{n_0-1} \in \overline{\mathfrak{N}}_\lambda$. But x_{n_0-1} is an arbitrary element of $\mathfrak{N}_{\lambda_0} \cap \mathfrak{M}_{\lambda_0}$, and, consequently, $\overline{\mathfrak{N}}_{\lambda_0} \cap \mathfrak{M}_{\lambda_0} \subset \overline{\mathfrak{N}}_\lambda$. Taking also into account that $\mathfrak{M}_{\lambda_0} \subset \mathfrak{M}_\lambda$, we obtain the inclusion $\overline{\mathfrak{N}}_{\lambda_0} \cap \mathfrak{M}_{\lambda_0} \subset \overline{\mathfrak{N}}_\lambda \cap \mathfrak{M}_\lambda$, valid for any λ from the circle $|\lambda - \lambda_0| < 1/2(N_{\lambda_0} + 1)$. If $\lambda_0 \in \Gamma$, then in the indicated circle $\overline{\mathfrak{N}}_{\lambda_0} \subset \overline{\mathfrak{N}}_\lambda$. Thus, the following holds.

Lemma 1. *Let λ_1, λ_2 be two points of $\Phi \setminus \Gamma$ such that*

$$|\lambda_1 - \lambda_2| < \frac{1}{2}(N_{\lambda_i} + 1), \quad i = 1, 2.$$

Then $\overline{\mathfrak{N}}_{\lambda_1} = \overline{\mathfrak{N}}_{\lambda_2}$ and $\mathfrak{M}_{\lambda_1} = \mathfrak{M}_{\lambda_2}$.

This result will be used below in the proof of Theorem 2.

Lemma 2. *Let $\lambda \in \Phi$ and $N_\lambda = \|\tilde{A}_\lambda^{-1}\|$. If F is a bounded closed set contained in $\Phi \setminus \Gamma$, then*

$$\sup_{\lambda \in F} N_\lambda < \infty.$$

Suppose the contrary. Then in F there is a convergent sequence $\{\lambda_n\}$ ($\lim \lambda_n = \lambda_0 \in F$) such that $\lim N_{\lambda_n} = \infty$. In the quotient space $\mathfrak{B}/\mathfrak{Z}_{A_{\lambda_n}}$, $n = 1, 2, \dots$, there exists an element \tilde{x}_n , $\|\tilde{x}_n\| = 1$, for which $\|\tilde{A}_{\lambda_n} \tilde{x}_n\| < 1/(N_{\lambda_n} - 1) \rightarrow 0$. Let $x_n \in \tilde{x}_n$ and $\|x_n\| < 2^{1/n}$. Since $\rho(x_n, \mathfrak{Z}_{A_{\lambda_0}}) \leq \|x_n\| < 2^{1/n}$, there exists a sequence $\{\delta_n\}$, $\delta_n > 0$, $\delta_n \rightarrow 0$, and a sequence $\{x_0^n\}$, $x_0^n \in \mathfrak{Z}_{A_{\lambda_0}}$ ($= \mathfrak{Z}_{A_{\lambda_0}} \subset \mathfrak{M}_{\lambda_0}$), satisfying the condition

$$\|x_n - x_0^n\| < \rho(x_n, \mathfrak{Z}_{A_{\lambda_0}}) + \delta_n < 2^{1/n}$$

(with this $\|x_0^n\| < 4$).

For each x_0^n construct an element $x_{\lambda_n}^n \in \mathfrak{Z}_{A_{\lambda_n}}$, setting

$$x_{\lambda_n}^n = x_0^n + (\lambda_n - \lambda_0)x_1^n + (\lambda_n - \lambda_0)^2 x_2^n + \dots,$$

where $A_{\lambda_0} x_{k+1}^n = x_k^n$, $\|x_{k+1}^n\| \leq (N_{\lambda_0} + 1)\|x_k^n\|$, $k = 0, 1, 2, \dots$. From the boundedness of the set $\{\|x_0^n\|\}$ it follows that $\|x_{\lambda_n}^n - x_0^n\| \rightarrow 0$. Since

$$\begin{aligned} \rho(x_n, \mathfrak{Z}_{A_{\lambda_0}}) + \delta_n &> \|x_n - x_0^n\| \geq \|x_n - x_{\lambda_n}^n\| - \|x_{\lambda_n}^n - x_0^n\| \geq \\ &\geq \rho(x_n, \mathfrak{Z}_{A_{\lambda_n}}) - \|x_{\lambda_n}^n - x_0^n\| = 1 - \|x_{\lambda_n}^n - x_0^n\|, \end{aligned}$$

we have $\rho(x_n, \mathfrak{Z}_{A_{\lambda_0}}) \rightarrow 1$. Let \tilde{x}_n denote the coset in $\mathfrak{B}/\mathfrak{Z}_{A_{\lambda_0}}$ containing x_n . Then $\|\tilde{x}_n\| = \rho(x_n, \mathfrak{Z}_{A_{\lambda_0}}) \rightarrow 1$, and $\|A_{\lambda_0} x_n\| = \|\tilde{A}_{\lambda_0} \tilde{x}_n\| \geq \|\tilde{x}_n\|/N_{\lambda_0} \rightarrow 1/N_{\lambda_0}$, which contradicts the equalities

$$\lim \|A_{\lambda_0} x_n\| = \lim \|A_{\lambda_n} x_n\| = \lim \|\tilde{A}_{\lambda_n} \tilde{x}_n\| = 0.$$

Remark. If $\mathfrak{Z}_{A_{1\lambda}} = \mathfrak{Z}_{A_\lambda}$ (which occurs at points $\lambda \in \Phi \setminus \Gamma$), then the operator $\tilde{A}_{1\lambda}$ is a restriction of the operator \tilde{A}_λ , whence $N_{1\lambda} \leq N_\lambda$. Therefore

$$\sup_{\lambda \in F} N_{1\lambda} \leq \sup_{\lambda \in F} N_\lambda < \infty.$$

Theorem 2. *If λ ranges over the set $G \setminus \Gamma$, where G is any connected component of the set Φ , then the spaces \mathfrak{R}_λ and \mathfrak{M}_λ are constant.*

Indeed, let λ_1, λ be any two points of $G \setminus \Gamma$, and let F be a polygonal line in $G \setminus \Gamma$ joining λ_1 to λ . Take on F a finite number of points λ_i , $i = 1, 2, \dots, m$; $\lambda_m = \lambda$, for which

$$|\lambda_i - \lambda_{i+1}| < 1/2(1 + \sup_{\lambda \in F} N_{1\lambda}).$$

Then (see Lemma 1)

$$\overline{\mathfrak{R}}_{\lambda_i} = \overline{\mathfrak{R}}_{\lambda_{i+1}}, \quad \mathfrak{M}_{\lambda_i} = \mathfrak{M}_{\lambda_{i+1}}, \quad i = 1, 2, \dots, m-1,$$

whence

$$\overline{\mathfrak{R}}_{\lambda_1} = \overline{\mathfrak{R}}_\lambda, \quad \mathfrak{M}_{\lambda_1} = \mathfrak{M}_\lambda.$$

As follows from the preceding, all elements belonging to \mathfrak{M}_λ ($\lambda \in G$) can be obtained by analytic continuation of power series generated by infinite chains of elements from \mathfrak{M}_{λ_0} (λ_0 is any fixed point in G). All elements obtained by continuation belong to the space of elements of the original chain and, in the case that G is simply connected, are single-valued analytic functions of the parameter λ .

Take the series considered above,

$$x_{0\lambda} = x_0 + (\lambda - \lambda_0)x_1 + (\lambda - \lambda_0)^2 x_2 + \dots$$

It can be continued to the whole component G and defines in it an analytic vector-valued function of λ , which is a zero of the operator A_λ . If one describes any closed curve in G (returning to the initial point λ_0), then as a result of analytic continuation along this curve we again obtain a zero of the operator A_{λ_0} . Since this zero must belong to the space determined by the elements x_0, x_1, x_2, \dots , it may differ from x_0 only by a numerical factor. Thus, whatever the component G of the set Φ , the zeros of the operator A_λ belonging to \mathfrak{M}_λ

are, up to a numerical factor, single-valued analytic functions in G . This is also true for the zero-elements of the operator A_λ , i.e. for all elements of \mathfrak{R}_λ .

Received
11 VIII 1963

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Note: Figure translations are in progress. See original paper for figures.

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