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Abstract

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ON A PARAMETRIC REPRESENTATION OF CERTAIN GENERAL CLASSES OF MEROMORPHIC FUNCTIONS IN THE UNIT DISK

One of the fundamental results in the theory of meromorphic functions is the well-known theorem of R. Nevanlinna ([1]) on the parametric representation of the class N, introduced by him, of meromorphic functions in the disk $|z| < 1$ with **bounded characteristic** (or, what is the same thing, the class of functions of **bounded type**).

An attempt to extend this theorem to meromorphic functions with unbounded characteristic was undertaken by us earlier ([2], [3]). At that time the classes N_α^* $(0 < \alpha < \infty)$ were introduced, consisting of functions $F(z)$, meromorphic in the disk $|z| < 1$, for which the characteristic function $T(r)$ satisfies the condition

$$\int_0^1 (1-r)^{\alpha-1} T(r)\, dr < +\infty.$$

However, the canonical representation of the class N_α^* obtained in that work did not have a parametric character, as is the case in R. Nevanlinna' s theorem for the class N.

In the present note a certain general class N_α of functions meromorphic in the disk $|z| < 1$, depending on an arbitrary parameter α $(-1 < \alpha < \infty)$, is defined, and a parametric representation of the class is established.

1°. We first recall the definitions of the fractional integral and the fractional derivative in the sense of Riemann—Liouville, which will be needed below.

Let the function $\varphi(r)$ be defined on $(0, 1)$ and belong to the class $L(0, 1)$. For any number $\alpha \in (0, +\infty)$, the **fractional integral of order** α of $\varphi(r)$ is the function

$$D^{-\alpha}\varphi(r) \equiv \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} \varphi(t)\, dt, \qquad r \in (0, 1), \tag{1}$$

where the operator $D^{-\alpha}\varphi(r)$ is defined almost everywhere on $(0, 1)$ and belongs to the class $L(0, 1)$.

It can be shown that almost everywhere on $(0,1)$

$$\lim_{\alpha\to+0} D^{-\alpha}\varphi(r)=\varphi(r). \tag{2}$$

Further, for any $\alpha\in(-1,0)$, the **fractional derivative of order** $\gamma=-\alpha$ of $\varphi(r)\in L(0,1)$ is customarily called the function

$$D^{-\alpha}\varphi(r)=\frac{d}{dr}D^{-(1+\alpha)}\varphi(r), \qquad r\in(0,1), \tag{3}$$

if it exists.

Finally, in view of (2) and (3), it is natural to identify both the integral and the derivative of zero order with the function itself. Let

$$D^{0}\varphi(r)\equiv\varphi(r). \tag{4}$$

Thus, the operator $D^{-\alpha}\varphi(r)$ is defined for an arbitrary value of the parameter α $(-1<\alpha<+\infty)$, provided, of course, that for values $\alpha\in(-1,0)$ it has meaning.

Lemma 1. Let the function

$$f\left(re^{i\varphi}\right)=\sum_{k=0}^{\infty}a_k\left(re^{i\varphi}\right)^k \tag{5}$$

be holomorphic in the unit disk. Then:

a) For any $\alpha\in(-1,+\infty)$ the function

$$f_\alpha\left(re^{i\varphi}\right)\equiv r^{-\alpha}D^{-\alpha}f\left(re^{i\varphi}\right)=\sum_{k=0}^{\infty}a_k\frac{\Gamma(1+k)}{\Gamma(1+\alpha+k)}\left(re^{i\varphi}\right)^k \tag{6}$$

is also holomorphic in the unit disk.

b) For any $\alpha\in(-1,+\infty)$ and $r\in(0,1)$ the integral formulas

$$f(z)=\frac{r}{2\pi}\int_0^{2\pi}\frac{\Gamma(1+\alpha)}{\left(r-ze^{-i\vartheta}\right)^{1+\alpha}}D^{-\alpha}f\left(re^{i\vartheta}\right)d\vartheta \qquad (|z|<r); \tag{7}$$

$$f(z)=-\overline{f(0)}+\frac{r}{\pi}\int_0^{2\pi}\frac{\Gamma(1+\alpha)}{\left(r-ze^{-i\vartheta}\right)^{1+\alpha}}D^{-\alpha}\operatorname{Re}f\left(re^{i\vartheta}\right)d\vartheta \qquad (|z|<r). \tag{8}$$

2°. Let the function $F(z)$ be meromorphic in the disk $|z| < 1$. Let $\{a_\mu\}$ and $\{b_\nu\}$ be, respectively, the sequences of its zeros and poles distinct from $z = 0$ and numbered in the order of nondecreasing moduli,

$$0 < |a_1| \leq |a_2| \leq \cdots \leq |a_\mu| \leq \cdots,$$

$$0 < |b_1| \leq |b_2| \leq \cdots \leq |b_\nu| \leq \cdots,$$

where we assume that each zero or pole is written as many times as its multiplicity.

Finally, if in a neighborhood of the origin there is a Laurent expansion $F(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots$ ($c_\lambda \neq 0$), then we shall regard the point $z = 0$ as a zero (for $\lambda \geq 1$) or a pole (for $\lambda \leq -1$) of multiplicity λ. With the aid of formula (8) of Lemma 1 one establishes

Lemma 2. For any $\alpha \in (-1, +\infty)$ and r $(0 < r < 1)$ the formula

$$\begin{aligned} \log F(z) = & -\log \overline{c_\lambda} - 2\lambda \psi_\alpha(r) + \lambda \log z \\ & + \sum_{|a_\mu| \leq r} \log \left\{ \left(1 - \frac{z}{a_\mu}\right) e^{-V_\alpha\left(\frac{z}{r}, \frac{a_\mu}{r}\right)} \right\} - \sum_{|b_\nu| \leq r} \log \left\{ \left(1 - \frac{z}{b_\nu}\right) e^{-V_\alpha\left(\frac{z}{r}, \frac{b_\nu}{r}\right)} \right\} \\ & + \frac{r}{\pi} \int_0^{2\pi} \frac{\Gamma(1+\alpha)}{\left(r - z e^{-i\vartheta}\right)^{1+\alpha}} D^{-\alpha} \log \left| F\left(r e^{i\vartheta}\right) \right| d\vartheta \qquad (|z| < r), \end{aligned} \tag{9}$$

where

$$\psi_\alpha(r) = \log r - \alpha \sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)},$$

$$\begin{aligned} V_\alpha(z; \zeta) = & \frac{1}{\pi} \int_0^{2\pi} \frac{\Gamma(1+\alpha)}{\left(1 - z e^{-i\vartheta}\right)^{1+\alpha}} \left\{ D^{-\alpha} \log \left| 1 - \frac{r e^{i\vartheta}}{\zeta} \right| \right\}_{r=1} d\vartheta - 2 \int_{|\zeta|}^1 \frac{(1-x)^\alpha}{x} \\ & - \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha+k)}{\Gamma(1+k)\Gamma(1+\alpha)} \left\{ \zeta^{-k} \int_0^{|\zeta|} (1-x)^\alpha x^{k-1}\, dx - \overline{\zeta}^k \int_{|\zeta|}^1 (1-x)^\alpha x^{-k-1}\, dx \right\} z^k. \end{aligned} \tag{10}$$

Let us note that in the case $\alpha = 0$ this formula is equivalent to the well-known Jensen–Nevanlinna formula, which lies at the basis of the theory of meromorphic functions.

Taking into account the asymptotic properties of the function $V_\alpha(z; \zeta)$, the following important theorem is proved.

Theorem 1. Let $\{z_k\}_1^\infty$ $(0 < |z_k| < 1)$ be any sequence of complex numbers, numbered in the order

$$0 < |z_1| \le |z_2| \le \cdots \le |z_k| \le \cdots.$$

and satisfying the condition

$$\sum_{k=1}^{\infty} (1 - |z_k|)^{1+\alpha} < +\infty, \tag{11}$$

where $\alpha \in (-1, +\infty)$. Then:

a) The infinite product

$$\pi_\alpha(z; z_k) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{-V_\alpha(z;z_k)} \tag{12}$$

converges uniformly and absolutely in every closed subdomain of the disk $|z| < 1$, representing an analytic function that vanishes only on the sequence $\{z_k\}_1^\infty$.

b) For any r $(0 < r < 1)$ and ϑ $(0 \le \vartheta \le 2\pi)$ we also have

$$D^{-\alpha} \log |\pi_\alpha(re^{i\vartheta}; z_k)| \le 0. \tag{13}$$

Let us further note that the function

$$B_\alpha(z; z_k) = \frac{\pi_\alpha(z; z_k)}{\sqrt{\pi_\alpha(0; z_k)}} = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{-\frac{1}{2} V_\alpha(z;z_k)} \tag{14}$$

is a natural analogue of the Blaschke product.

This should be understood in the sense that if

$$\sum_{k=1}^{\infty} (1 - |z_k|) < +\infty, \tag{11'}$$

then we shall have

$$\lim_{\alpha \to +0} B_\alpha(z; z_k) = B(z; z_k) = \prod_{k=1}^{\infty} \frac{z_k - z}{1 - \overline{z_k} z} \cdot \frac{|z_k|}{z_k}. \tag{14'}$$

3°. We now pass to the definition of the classes of meromorphic functions N_α. Let $n(t, 0)$ and $n(t, \infty)$ be, respectively, the number of numbers $\{a_\mu\}$ and $\{b_\nu\}$

lying in the disk $|z| \leq t$ $(0 < t < 1)$. We shall further agree that $n(0,0)$ and $n(0,\infty)$ denote, respectively, the multiplicity of the zero or pole of the function $F(z)$ at the origin $z = 0$, so that we shall have

$$n(0,0) - n(0,\infty) = \lambda.$$

Introduce the functions

$$N_\alpha(r;0) = \frac{r^{-\alpha}}{\Gamma(1+\alpha)} \int_0^r \frac{(r-t)^\alpha}{t} [n(t,0) - n(0,0)]\, dt + \frac{n(0,0)}{\Gamma(1+\alpha)}[\log r - K_\alpha],$$

$$N_\alpha(r;\infty) = \frac{r^{-\alpha}}{\Gamma(1+\alpha)} \int_0^r \frac{(r-t)^\alpha}{t} [n(t,\infty) - n(0,\infty)]\, dt + \tag{15}$$

$$+ \frac{n(0,\infty)}{\Gamma(1+\alpha)}[\log r - K_\alpha],$$

where the value of the parameter $\alpha \in (-1,+\infty)$ is arbitrary, and

$$K_\alpha = -\psi_\alpha(1) = \alpha \sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)}.$$

Next, setting

$$D_{(+)}^{-\alpha}\varphi(r) = \begin{cases} D^{-\alpha}\varphi(r), & \text{if } D^{-\alpha}\varphi(r) \geq 0, \\ 0, & \text{if } D^{-\alpha}\varphi(r) \leq 0, \end{cases} \tag{16}$$

and also

$$D_{(-)}^{-\alpha}\varphi(r) = D_{(+)}^{-\alpha}\varphi(r) - D^{-\alpha}\varphi(r), \tag{16'}$$

denote

$$m_\alpha(r,\infty) = \frac{r^{-\alpha}}{2\pi} \int_0^{2\pi} D_{(+)}^{-\alpha} \log\left|F\left(re^{i\vartheta}\right)\right|\, d\vartheta,$$

$$m_\alpha(r,0) = \frac{r^{-\alpha}}{2\pi} \int_0^{2\pi} D_{(-)}^{-\alpha} \log\left|F\left(re^{i\vartheta}\right)\right|\, d\vartheta. \tag{17}$$

Lemma 3. a) For any $\alpha \in (-1,+\infty)$ the identity

$$m_\alpha(r,\infty)+N_\alpha(r,\infty)=$$

$$=m_\alpha(r,0)+N_\alpha(r,0)+\frac{\log|C_\lambda|}{\Gamma(1+\alpha)}\quad(0<r<1) \tag{18}$$

holds.

b) The function

$$T_\alpha(r)\equiv m_\alpha(r,\infty)+N_\alpha(r,\infty)\quad(0<r<1) \tag{19}$$

is monotonically increasing.

We shall call the function $T_\alpha(r)$ the α-characteristic of the meromorphic function $F(z)$. Finally, let N_α denote the set of all functions $F(z)$ meromorphic in the disk $|z|<1$ for which, for a given $\alpha\in(-1,+\infty)$, the α-characteristic is bounded, i.e., for which

$$T_\alpha(1)=\lim_{r\to1-0}T_\alpha(r)<+\infty. \tag{20}$$

From the definition (19) of the function $T_\alpha(r)$ and the class N_α, in view of (4), it is obvious that for the value of the parameter $\alpha=0$ we shall have

$$T_0(r)\equiv T(r),\qquad N_0\equiv N, \tag{21}$$

where $T(r)$ is the characteristic, and N is the class of functions with bounded characteristic, introduced by R. Nevanlinna.

It has been established that the inclusion

$$N_{\alpha_1}\subset N_{\alpha_2}\quad\text{for }-1<\alpha_1<\alpha_2<+\infty, \tag{22}$$

holds, and therefore also the inclusions

$$N_\alpha\subset N_0\quad\text{for }-1<\alpha\le0;\qquad N_0\subset N_\alpha\quad\text{for }0\le\alpha<+\infty. \tag{22'}$$

Thus, we have a family $\{N_\alpha\}$, depending on the parameter α $(-1<\alpha<+\infty)$, of classes of functions meromorphic in the disk $|z|<1$. These classes possess property (22), i.e. they expand as α increases, and the known class N, introduced by R. Nevanlinna, is contained in our family $\{N_\alpha\}$ for the value of the parameter $\alpha=0$.

Despite the considerable generality of the family of classes $\{N_\alpha\}$, the theorem on their parametric representation has the same complete character as R. Nevanlinna' s theorem on the class $N \equiv N_0$.

Theorem 2. The class N_α $(-1 < \alpha < \infty)$ coincides with the set of functions admitting in the disk $|z| < 1$ a representation of the form

$$F(z) = A_F z^\lambda \frac{\pi_\alpha(z; a_\mu)}{\pi_\alpha(z; b_\nu)} \exp\left\{\frac{1}{\pi}\int_0^{2\pi} \frac{d\psi(\theta)}{(1 - e^{-i\theta}z)^{1+\alpha}}\right\} \quad (|z| < 1), \tag{23}$$

where $\pi_\alpha(z; a_\mu)$ and $\pi_\alpha(z; b_\nu)$ are convergent products of the form (12), $\psi(\theta)$ is an arbitrary function of bounded variation on $[0, 2\pi]$, and, finally, A_F is an arbitrary constant.

This theorem, in particular, contains the well-known theorem of R. Nevanlinna on the parametric representation of the class $N \equiv N_0$.

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Note: Figure translations are in progress. See original paper for figures.