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Abstract

Full Text

MATHEMATICS

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ON HYPOELLIPTIC EQUATIONS IN THE WHOLE SPACE AND THE LIMITING ABSORPTION PRINCIPLE

(Presented by Academician G. I. Petrov, 10 XI 1963)

In papers ⁽¹⁻³⁾ we considered the question of what conditions at infinity can single out a class of functions W in which there exists a unique solution in the whole space of the equation

$$P\left(\frac{\partial}{\partial x}\right)u(x) = f(x), \quad (1)$$

where $P(i\partial/\partial x) = P(i\partial/\partial x_1, \dots, i\partial/\partial x_n)$ is a hypoelliptic operator ⁽⁴⁾ with constant coefficients, $P(\sigma) = P(\sigma_1, \dots, \sigma_n)$ is its characteristic polynomial, and $f(x)$ is an arbitrary finite function (possibly a generalized one).

In § 1 of the present paper we indicate one more method for singling out classes W (by means of conditions at infinity in integral form); in § 2 we prove the applicability to equation (1) of the limiting absorption principle and determine what conditions at infinity can be used to single out the solution of equation (1) obtained by means of the limiting absorption principle. In § 3 we generalize the results of ⁽¹⁻³⁾ to the case when the operator $P(i\partial/\partial x)$ has variable coefficients.

Let the operator $P(i\partial/\partial x)$ satisfy the following three conditions:

1. The dimension of the real zeros of $P(\sigma)$ is equal to $n - 1$.
2. $\text{grad } P(\sigma) \neq 0$ at the real zeros of $P(\sigma)$.

From these conditions it follows that the real zeros of the polynomial $P(\sigma)$ form several smooth closed surfaces of dimension $n - 1$. Denote them by K_j , $j = 1, 2, \dots, t$.

3. The total curvature of the surfaces K_j is positive at every point.

Let $x = r\omega$, where $r = (\sum x_i^2)^{1/2}$, ω is a unit vector, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, $\omega_i = x_i/r$. On each surface K_j choose an arbitrary orientation (specify the direction of the normal). This can be done in 2^t different ways. Then on each surface K_j , $j = 1, 2, \dots, t$, there exists exactly one point $\sigma_j(\omega)$ at which the normal vector to the surface K_j is parallel to the vector ω and has the same

direction. By $\mu_j(\omega)$ we denote the magnitude of the projection of $\sigma_j(\omega)$ onto the vector ω :

$$\mu_j(\omega) = (\sigma_j(\omega), \omega) = \sum_{i=1}^n \sigma_{ij}(\omega) \omega_i.$$

Let $p = (p_1, \dots, p_n)$ be an integer vector, $p_i \geq 0$, $|p| = \sum p_i$,

$$D_p \left(\frac{\partial}{\partial x} \right) = \frac{\partial^{|p|}}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}, \quad D_p(\sigma) = \sigma_1^{p_1} \dots \sigma_n^{p_n}.$$

Recall that in papers (1-3) we obtained the following two theorems:

Theorem A. For any hypoelliptic operator $P(i\partial/\partial x)$ satisfying conditions 1-3, there exists a fundamental solution $E(x)$ which, as $r \rightarrow \infty$, has the following asymptotic representation:

$$D_p \left(\frac{\partial}{\partial x} \right) E(x) = \sum_{j=1}^t A_j(\omega) D(-i\sigma_j(\omega)) e^{-i\mu_j(\omega)r} r^{(1-n)/2} + E_p(x), \quad (2)$$

where $A_j(\omega)$ is some infinitely differentiable function and, in a neighborhood of infinity,

$$|E_p(x)| < C(p) r^{-n/2}. \quad (3)$$

Theorem B. If $P(i\partial/\partial x)$ is a hypoelliptic operator satisfying conditions 1-3, and $f(x)$ is any finite function, then equation (1) has a unique solution in the following class of functions W : $u(x) \in W$, if it can be represented in the form of a sum of functions $u(x) = \sum_{j=1}^t u_j(x)$, for which, in a neighborhood of infinity, the inequalities

$$|u_j(x)| < Cr^{(1-n)/2}; \quad \left| \frac{\partial u_j(x)}{\partial r} + i\mu_j(\omega) u_j(x) \right| < Cr^{-n/2}. \quad (4)$$

hold.

Let us note that, since the set of functions $\mu_j(\omega)$ can be chosen in 2^t ways, Theorems A and B give 2^t different fundamental solutions of the operator $P(i\partial/\partial x)$ and, correspondingly, classes W .

§ 1. Denote by $I_R[v(x)]$ the integral $R^{1-n} \int_{R < r < 2R} v^2(x) dx$. In exactly the same way as Theorem B, the following is proved.

Theorem 1. If $P(i\partial/\partial x)$ is a hypoelliptic operator satisfying conditions 1-3, and $f(x)$ is an arbitrary finite function, then equation (1) has a unique solution

in the class of functions $W: u(x) \in W$, if it can be represented in the form of a sum of functions $u(x) = \sum_{j=1}^t u_j(x)$, for which, as $R \rightarrow \infty$, the estimates

$$I_R[u_j(x)] < C; \quad I_R \left[\frac{\partial u_j(x)}{\partial r} + i\mu_j(\omega) u_j(x) \right] < \frac{C}{R} \quad (5)$$

hold.

It is clear that these conditions will single out the same solution of equation (1) as the conditions given by Theorem B.

§ 2. A polynomial satisfying conditions 1-3 can always be represented in the form $P(\sigma) = P_1(\sigma)P_2(\sigma)$, where $P_1(\sigma)$ is a polynomial with real coefficients, while the polynomial $P_2(\sigma)$ has no real zeros. Denote by $P_\varepsilon(i\partial/\partial x)$ the operator with characteristic polynomial $P_\varepsilon(\sigma) = [P_1(\sigma) + \varepsilon]P_2(\sigma)$. We shall take only such $\varepsilon = \varepsilon_1 + i\varepsilon_2$ for which $\varepsilon_2 \neq 0$. Then the polynomial $P_\varepsilon(\sigma)$ no longer has real zeros, and the equation

$$P_\varepsilon(i\partial/\partial x) u_\varepsilon(x) = f(x) \quad (6)$$

will have a unique solution, for example, in the class of functions L decreasing as $r \rightarrow \infty$ ⁽⁵⁾. Obviously, in order to find $\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x)$, it is enough to determine how, as $\varepsilon \rightarrow 0$, the fundamental solutions of the operator $P_\varepsilon(i\partial/\partial x)$ behave. Denote by $E_\varepsilon(x)$ the fundamental solution of the operator $P_\varepsilon(i\partial/\partial x)$ belonging to the class L . It is unique. This fundamental solution is given by the integral

$$E_\varepsilon(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \frac{e^{-i(x,\sigma)}}{P_\varepsilon(\sigma)} d\sigma. \quad (7)$$

We shall say that $\varepsilon \rightarrow \pm 0$ if $\varepsilon \rightarrow 0$ in such a way that $\varepsilon_2 \rightarrow \pm 0$ and the ratio $|\varepsilon_1/\varepsilon_2|$ remains greater than some positive constant at all times. Let us call natural those two of the 2^t fundamental solutions constructed in Theorem A which are obtained if, in orienting the surfaces K_j , the normal vector is taken simultaneously, for all K_j , as ...

$j = 1, 2, \dots, t$, or the vector $\text{grad} P_1(\sigma)$, or the opposite vector. We shall call natural the corresponding classes W given either by Theorem B or by Theorem 1.

The fundamental solutions for the operator $P(i\partial/\partial x)$ discussed in Theorem A were constructed by us in the form of an integral over a certain set H , located in the complex space $R_n(s)$, $s = \sigma + i\tau$:

$$E(x) = \frac{1}{(2\pi)^n} \int_H \frac{e^{-i(x,s)}}{P(s)} ds.$$

From this integral representation for $E(x)$ and formula (7) one can obtain the following assertion:

Theorem 2. *As $\varepsilon \rightarrow +0$, $E_\varepsilon(x)$ converges in the weak sense to one of the natural fundamental solutions of the operator $P(i\partial/\partial x)$, and as $\varepsilon \rightarrow -0$, to another.*

From this, various theorems on the convergence of a solution of equation (6) to a solution of equation (1) follow at once. For example:

Theorem 3. *For any generalized finite function $f(x)$, as $\varepsilon \rightarrow \pm 0$ the solution of equation (6), belonging to L , converges in the weak sense to a solution of equation (1), belonging to one of the natural classes W . Moreover, if $P(i\partial/\partial x)$ is an elliptic operator of order $2m$ and $f(x) \in W_p^k$ ($p > 1$), then on every compact set $u_\varepsilon(x) \rightarrow u(x)$ in the space W_p^{2m+k} ; if $P(i\partial/\partial x)$ is an elliptic operator and $f(x) \in C^\alpha$ ($\alpha > 0$), then on every compact set $u_\varepsilon(x) \rightarrow u(x)$ in $C^{2m+\alpha}$.*

For questions concerning the convergence of $u_\varepsilon(x)$ to $u(x)$, the following is also useful.

Theorem 4. *For any p , the functions $D_p E_\varepsilon(x)$ are uniformly bounded for $|\varepsilon| \leq 1$ on every set not containing the origin.*

§ 3. We now consider operators with variable coefficients of the form

$$P\left(x, i\frac{\partial}{\partial x}\right) = P_0\left(i\frac{\partial}{\partial x}\right) + \lambda P_1\left(x, i\frac{\partial}{\partial x}\right), \quad (8)$$

where $P_0(i\partial/\partial x)$ is an elliptic operator of order $2m$ with constant coefficients satisfying conditions 1-3, and $P_1(x, i\partial/\partial x)$ is any operator whose coefficients are finite functions. It is assumed that the coefficients of the operator $P_1(x, i\partial/\partial x)$ standing at derivatives of order j have j continuous derivatives. A fundamental solution for the operator $P(x, i\partial/\partial x)$ will be a function $E(x, y)$ for which $P(x, i\partial/\partial x)E(x, y) = \delta(x - y)$. Let, as before, $r = (\sum x_i^2)^{1/2}$, $\omega_i = x_i/r$, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$. By Ω we denote the set of all values λ for which the operator $P(x, i\partial/\partial x)$ is elliptic. If the order of the operator $P_1(x, i\partial/\partial x)$ is less than $2m$, then Ω is the whole complex λ -plane. By Ω_0 we denote that connected component of the set Ω which contains the point $\lambda = 0$.

Theorem 5. *For all λ in Ω_0 , with the possible exception of a certain discrete set Λ_1 , the operator $P(x, i\partial/\partial x)$ has a fundamental solution $E(x, y)$ possessing the following asymptotic representation as $r \rightarrow \infty$:*

$$D_p\left(\frac{\partial}{\partial x}\right)E(x, y) = \sum_{j=1}^t A_j(\omega, y)D(-i\sigma_j(\omega))e^{-i\psi_j(\omega)r}r^{(1-n)/2} + E_p(x, y),$$

where $A_j(\omega, y)$ is a continuous function of the arguments ω, y , and as $r \rightarrow \infty$, $|y| < a$,

$$|E_p(x, y)| < C(p, a) r^{-n/2}.$$

Denote by Λ the discrete set belonging to Ω_0 and consisting of Λ_1 and the analogous set Λ_2 , constructed for the operator $\bar{P}^*(x, i\partial/\partial x)$.

Exactly as Theorems B and 1 followed from Theorem A, the following assertion can now be obtained from Theorem 5:

Theorem 6. For all λ from $\Omega_0 \setminus \Lambda$ and any finite function $f(x)$ from L_p ($p > 1$), the equation

$$P(x, i\partial/\partial x)u(x) = f(x) \quad (9)$$

has a unique solution in the class W : $u(x) \in W$, if it is representable in the form of a sum of functions

$$u(x) = \sum_{j=1}^t u_j(x),$$

for which, in a neighborhood of infinity, either inequalities (4) or inequalities (5) hold.

It is not difficult to give an example in which the set Λ is nonempty. Let

$$P_0(i\partial/\partial x) = \Delta + k^2, \quad x = (x_1, x_2, x_3).$$

The operator $P_1(x, i\partial/\partial x)$ will simply be multiplication by some finite infinitely differentiable function $\psi(x)$. We shall find such a $\psi(x)$ that the equation

$$\Delta u + k^2 u + \psi(x)u = 0$$

has a nonzero solution in the class of functions for which, as $r \rightarrow \infty$,

$$|u(x)| < Cr^{-1}, \quad |\partial u/\partial r - iku| < Cr^{-2}.$$

This will show that $\lambda = 1$ belongs to Λ . Take the function $u(x)$ equal to e^{ikr}/r for $r \geq 1$, and extend it inside the disk $r < 1$ in an arbitrary way so that an infinitely differentiable function is obtained which nowhere vanishes. This can be done. Then

$$\Delta u + k^2 u = \varphi(x),$$

where $\varphi(x) = 0$ for $r \geq 1$. As $\psi(x)$ take the function $-\varphi(x)/u(x)$.

We shall now give one sufficient condition for $\lambda_0 \notin \Lambda$. Suppose that for some $\lambda = \lambda_0 \neq 0$ the operator $P(x, i\partial/\partial x)$ is formally self-adjoint. Then $\Omega_0 = \Omega$ and the following holds.

Theorem 7. Suppose that for some $\lambda = \lambda_0 \neq 0$: a) the operator $P(x, i\partial/\partial x)$ is formally self-adjoint; b) each of the irreducible factors of the polynomial $P_0(\sigma)$ has real zeros; c) the equation $P(x, i\partial/\partial x)u(x) = 0$ has no finite solutions (for sufficient conditions for this, see (6)).

Then for $\lambda = \lambda_0$ the operator (8) has natural fundamental solutions, and for any finite function $f(x)$ from L_p ($p > 1$) equation (9) has a unique solution in any of the natural classes W .

Remark 1. A theorem analogous to Theorem B was obtained independently of the author by V. V. Grushin ⁽⁷⁾. In V. V. Grushin's work the classes W were distinguished not by conditions (4), but by a certain differential inequality of high order on the function $u(x)$:

$$|u(x)| < Cr^{(1-n)/2}, \quad |Q(\omega, i\partial/\partial x)u(x)| < Cr^{-n/2}. \quad (10)$$

In the corresponding formulation he also obtained a theorem analogous to Theorem 1. Theorems 3, 6, 7 remain valid when conditions (4), (5) are replaced by the corresponding conditions of the form (10).

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Note: Figure translations are in progress. See original paper for figures.

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