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Abstract

Full Text

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A CRITERION FOR THE FINITENESS OF THE NUMBER OF DISCRETE LEVELS INTRODUCED INTO THE GAPS OF THE CONTINUOUS SPECTRUM BY PERTURBATIONS OF A PERIODIC POTENTIAL

(Presented by Academician V. I. Smirnov on 15 I 1964)

It is known that the spectrum of the problems

$$y'' + \{\lambda - q(x)\}y = 0, \quad -\infty < x < \infty, \quad (1)$$

with a real periodic potential $q(x) = q(x + 1)$, is purely continuous, is bounded below, and consists of a sequence of intervals extending to $+\infty$, separated by a finite or, in general, infinite number of gaps ⁽¹⁾. It is also known ⁽²⁾ that the perturbed problem

$$y'' + \{\lambda - q(x) - p(x)\}y = 0, \quad -\infty < x < \infty, \quad (2)$$

in the case of a real nonperiodic perturbation $p(x)$, which in one sense or another is sufficiently small as $x \rightarrow \pm\infty$ (in particular, under condition (3)), has the same continuous spectrum as problem (1), and in each of the gaps there may appear a finite or infinite number of discrete eigenvalues, which can accumulate only at the ends of the gaps.

In a report by M. Sh. Birman and I. M. Glazman at the Fourth All-Union Mathematical Congress in 1961 ⁽³⁾ (see also ⁽²⁾, Ch. V, § 56), the question was posed of conditions for the finiteness of the number of discrete levels introduced into the gaps by the perturbation. The following theorem is devoted to this question.

Theorem. *Under the condition*

$$\int_{-\infty}^{\infty} (1 + |x|) |p(x)| dx < \infty \quad (3)$$

in each of the gaps of the continuous spectrum of the self-adjoint problem (2), where $q(x)$ is a periodic function, there appears no more than a finite number

of eigenvalues; moreover, sufficiently distant gaps contain, in any case, no more than two eigenvalues each.

Eigenvalues cannot be superimposed on the continuous spectrum and, in particular, the ends of the gaps cannot turn out to be eigenvalues.

Proof*. We note the following facts. Let $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ be solutions of the unperturbed equation (1) under the initial conditions

$$\theta(0, \lambda) = 1, \quad \theta'(0, \lambda) = 0; \quad \varphi(0, \lambda) = 0, \quad \varphi'(0, \lambda) = 1.$$

The multipliers of equation (1) are the roots of the equation

$$\rho^2 - [\theta(1, \lambda) + \varphi'(1, \lambda)]\rho + 1 = 0.$$

Here $\rho_1 = \rho^{-1}(\lambda)$, $\rho_2 = \rho(\lambda)$, where $|\rho(\lambda)| \geq 1$, and the equality sign is attained on the spectrum and only on the spectrum (see, for example, ^(1,4)). The functions $\rho_j(\lambda) \neq 0$ are regular and single-valued in the whole λ -plane with cuts along the spectrum. For $\rho(\lambda) \neq \pm 1$, equation (1) has a fundamental system of solutions of the form

$$e_1(x, \lambda) = e^{-k(\lambda)x} z_1(x, \lambda), \quad e_2(x, \lambda) = e^{k(\lambda)x} z_2(x, \lambda),$$

* The finiteness of the number of discrete levels to the left of the lower edge of the continuous spectrum under the conditions of the theorem is known ⁽²⁾.

where it is put that $k(\lambda) = \alpha(\lambda) + i\beta(\lambda) = \ln \rho(\lambda)$, and $z_j(x, \lambda)$ ($j = 1, 2$) are functions periodic in x : $z_j(x+1, \lambda) = z_j(x, \lambda)$ (multi-valued in λ). The solutions $e_j(x, \lambda)$ are analytic in λ , single-valued in the λ -plane cut along the spectrum, and can be defined by the formula

$$e_j(x, \lambda) = \varphi(1, \lambda)\theta(x, \lambda) + [\rho_j(\lambda) - \theta(1, \lambda)]\varphi(x, \lambda) \quad (j = 1, 2), \quad (4)$$

where $e_j(x+1, \lambda) = \rho_j(\lambda)e_j(x, \lambda)$. (When $\varphi(1, \lambda) = 0$, one of the solutions $e_j(x, \lambda)$ is identically zero in x , and its normalization in this case must be changed.) For $\rho(\lambda) = \pm 1$ the solutions $e_j(x, \lambda)$ coincide: $e_1(x, \lambda) = e_2(x, \lambda)$, and are periodic, while the remaining solutions grow no faster than linearly in x .

Lemma 1. If the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} p & r \\ 0 & s \end{pmatrix}$ are unitarily equivalent, then $|r| \leq |b| + |c|$.

Lemma 2. For any solution $u(x, \lambda)$ of equation (1) with periodic potential $q(x) = q(x+1)$, for all complex λ and real x and t the estimate holds

$$|u(x, \lambda)| + \frac{|u'(x, \lambda)|}{1 + |\sqrt{\lambda}|} \leq C e^{\alpha(\lambda)|x-t|} (1 + |x-t|) \left(|u(t, \lambda)| + \frac{|u'(t, \lambda)|}{1 + |\sqrt{\lambda}|} \right), \quad (5)$$

where $\alpha(\lambda) = \ln |\rho(\lambda)| \geq 0$.

The lemma is also valid for complex potentials.

Proof of Lemma 2. Let

$$\mathbf{h}(x, \lambda) = \left\{ u(x, \lambda); \frac{u'(x, \lambda)}{1 + |\sqrt{\lambda}|} \right\}$$

be a column vector. Then

$$\mathbf{h}(n, \lambda) = B^n(\lambda) \mathbf{h}(0, \lambda),$$

where $n = 0, 1, 2, \dots$, and the matrix

$$B(\lambda) = \begin{pmatrix} \theta(1, \lambda) & (1 + |\sqrt{\lambda}|)\varphi(1, \lambda) \\ (1 + |\sqrt{\lambda}|)^{-1}\theta'(1, \lambda) & \varphi'(1, \lambda) \end{pmatrix}$$

has eigenvalues $\rho(\lambda)$ and $\rho^{-1}(\lambda)$. By a unitary transformation the matrix $B^n(\lambda)$ is brought to the form

$$\begin{pmatrix} \rho^n(\lambda) & r_n(\lambda) \\ 0 & \rho^{-n}(\lambda) \end{pmatrix},$$

where, by Lemma 1,

$$|r_n(\lambda)| \leq n|\rho(\lambda)|^{n-1} \left\{ \frac{|\theta'(1, \lambda)|}{1 + |\sqrt{\lambda}|} + (1 + |\sqrt{\lambda}|)|\varphi(1, \lambda)| \right\} \leq Cn|\rho(\lambda)|^n.$$

Therefore $|\mathbf{h}(n, \lambda)| \leq C(n+1)|\rho(\lambda)|^n |\mathbf{h}(0, \lambda)|$. Taking into account the asymptotics of $\rho(\lambda)$ as $|\lambda| \rightarrow \infty$, it is not difficult to extend this estimate to intermediate values of x . The estimate for $x < 0$ is carried out in the same way. Finally, owing to the periodicity of $q(x)$, the constant C in formula (5) can be chosen independent also of t . Lemma 2 is proved.

Lemma 3. Under condition (3), the perturbed equation (2) has solutions of the form

$$\begin{aligned}
 E_1(x, \lambda) &= e_1(x, \lambda) - \int_x^\infty K(x, t, \lambda)p(t)e_1(t, \lambda) dt, \\
 E_2(x, \lambda) &= e_2(x, \lambda) + \int_{-\infty}^x K(x, t, \lambda)p(t)e_2(t, \lambda) dt.
 \end{aligned}
 \tag{6}$$

For each x , the functions $E_j(x, \lambda)$ are analytic in λ in the plane of problem (1) cut along the spectrum and are continuous up to the cuts, inclusive. The kernel $K(x, t, \lambda)$ is an entire function of λ for every pair of values x, t , admits the estimate

$$|K(x, t, \lambda)| \leq C \frac{1 + |x - t|}{1 + |\sqrt{\lambda}|} e^{\alpha(\lambda)|x-t|}
 \tag{7}$$

and is the Cauchy function of equation (2), i.e., it satisfies equation (2) in x ($-\infty < x < \infty$), with $K(t, t, \lambda) = 0$, $K'_x(x, t, \lambda)|_{x=t} = 1$. The lemma is also valid for complex $q(x)$ and $p(x)$.

Proof of Lemma 3. The kernel $K(x, t, \lambda)$ is connected with the equation

$$K(x, t, \lambda) = C(x, t, \lambda) + \int_t^x C(x, s, \lambda)p(s)K(s, t, \lambda) ds
 \tag{8}$$

with the Cauchy function $C(x, t, \lambda)$ of the unperturbed equation (1). Taking into account that $C(x, t, \lambda)$ satisfies equation (1) in x and the initial conditions $C(t, t, \lambda) = 0$, $C'_x(x, t, \lambda)|_{x=t} = 1$, we apply to $C(x, t, \lambda)$ estimate (5). Then, solving equation (8) by the method of successive approximations, we obtain for $K(x, t, \lambda)$ estimate (7) and verify the validity of the remaining assertions of Lemma 3.

The proof of the theorem is now completed with the aid of the splitting method due to I. M. Glazman ⁽²⁾. Consider the auxiliary operators L_1, L_2, L_3 , generated by equation (2): $l[y] = \lambda y$, respectively on the intervals $(-\infty, -M')$, $(-M', M)$, (M, ∞) with boundary conditions $y(-M') = y(M) = 0$. The continuous spectrum of the operators L_1 and L_3 is the same as for problems (1) and (2). If some interval Δ of the λ -axis: $(\lambda_0 - \delta, \lambda_0 + \delta)$ contains no more than n points of the spectrum of the operators L_1, L_2 and L_3 (counting multiplicities), then problem (2) has no more than $n + 2$ eigenvalues in the interval Δ . Otherwise, from the corresponding eigenfunctions of problem (2) one could construct a linear manifold of functions of dimension greater than n , vanishing at the points $-M'$ and M , for which

$$\int_{-\infty}^{\infty} |l[y] - \lambda_0 y|^2 dx < \delta^2 \int_{-\infty}^{\infty} |y|^2 dx.$$

This would mean that the operator

$$\tilde{L} \equiv L_1 \oplus L_2 \oplus L_3$$

has more than n eigenvalues $\lambda_i \in \Delta$, contrary to the assumption.

Let us now choose in an appropriate way the splitting points $-M'$ and M , separately for each gap under consideration. We note that, using the asymptotics of the solutions $e_j(x, \lambda)$ (4) and some additional considerations, one can show that for each gap $(\lambda'_k, \lambda''_k)$ there will be numbers ε_{jk} , $0 \leq \varepsilon_{jk} \leq 1$, such that, for some $\gamma > 0$ fixed for all gaps,

$$|e_j(\varepsilon_{jk}, \lambda)| \geq \gamma \max_{0 \leq x \leq 1} |e_j(x, \lambda)| > 0 \quad (j = 1, 2; \lambda'_k \leq \lambda \leq \lambda''_k).$$

Hence, and from Lemma 3, it follows that for a fixed sufficiently large integer N the functions $E_1(N + \varepsilon_{1k}, \lambda)$ and $E_2(-N + \varepsilon_{2k}, \lambda)$ have no zeros in the corresponding gaps $\lambda'_k \leq \lambda \leq \lambda''_k$ ($k = 0, 1, 2, \dots$), i.e., in these gaps there are no eigenvalues of the auxiliary problems for the intervals $(N + \varepsilon_{1k}, \infty)$, $(-\infty, -N + \varepsilon_{2k})$ of the x -axis. Finally, in each gap there falls no more than a finite number of eigenvalues of the regular problem for equation (2) on the interval $(-N + \varepsilon_{2k}, N + \varepsilon_{1k})$. The lengths of the gaps tend to zero as $\lambda \rightarrow +\infty$ (¹). Therefore it becomes possible, by carefully varying the numbers ε_{jk} , to free each of the sufficiently distant gaps completely from eigenvalues of all three auxiliary problems corresponding to it. This proves the theorem.

Let us note, incidentally, that the eigenvalues of problem (2) are the roots of the Wronskian determinant for the solutions $E_1(x, \lambda)$ and $E_2(x, \lambda)$. Let us also note that, in the case of finite perturbations, the finiteness of the number of discrete levels arising in a gap is a direct consequence of the splitting method and holds for self-adjoint problems of any order, not necessarily with periodic coefficients.

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REFERENCES

1. E. C. Titchmarsh, *Eigenfunction Expansions*, 2, Moscow, 1961.

2. I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Moscow–Leningrad, 1964.
3. M. Sh. Birman, I. M. Glazman, Spectra of singular differential operators. Proceedings of the IV All-Union Mathematical Congress, 2, Leningrad, 1964.
4. F. S. Rofe-Beketov, DAN, 152, No. 6 (1963).

Note: Figure translations are in progress. See original paper for figures.

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