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Abstract

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V. M. Volosov, B. I. Morgunov

Stationary Resonance Regimes of Certain Oscillatory Systems

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§ 1. **Statement of the problem.** In papers ($^{1-5}$), oscillatory systems with one degree of freedom containing slowly varying parameters were considered, for which the perturbation does not depend explicitly on time. For such systems, in the first and second approximations, amplitude curves and a number of other parameters characterizing the perturbed motion were found. However, those papers did not consider the cases in which the perturbation depends explicitly on time, in particular the resonance cases important for applications. In the present paper, stationary resonance regimes are investigated for the case in which the perturbation depends periodically on time. Analogous problems for rotational motions were considered in (^{8}).

Let the oscillatory motion be described by a system of the form

$$\frac{d}{dt}\left[m(x)\dot{y}\right] + Q(x, y) = \varepsilon f(x, y, \dot{y}, \vartheta),$$

$$\dot{x} = \varepsilon X(x, y, \dot{y}, \vartheta), \qquad \dot{\vartheta} = \nu(x) + \varepsilon\Theta(x, y, \dot{y}, \vartheta). \tag{1}$$

Here y is a one-dimensional coordinate; $m(x)$ is the mass, $x = (x_1, \dots, x_n)$ is the set of slowly varying parameters; $Q(x, y) \equiv \partial V(x, y)/\partial y$ is the potential force producing the oscillations; ϑ is the phase of the external perturbing force; f is a small nonlinear perturbation. The functions f, X, Θ are assumed to be periodic in ϑ with period 2π. We pose the problem: to find, in the first approximation in ε ($\varepsilon > 0$ is a small parameter), the coordinates of stationary resonance regimes and to obtain sufficient conditions for the stability (in a certain generalized sense) of these regimes.

§ 2. **Principal results.** On the basis of the knowledge of the integrals of the unperturbed system (the system into which (1) passes for $\varepsilon = 0$), we pass in (1) from the variables y and $\dot{y}$ to F_1, F_2, ψ, where F_1 and F_2 are respectively the maximum and minimum values of y, and ψ is the phase of the natural oscillations. (Analogous substitutions were carried out, for example, in ($^{1-5}$).) As a result we arrive at a system of the form

$$\dot{F}_1 = \varepsilon A(x, F_1, F_2, y, \dot{y}, \vartheta),$$

$$\dot{x} = \varepsilon X(x, y, \dot{y}, \vartheta), \tag{2}$$

$$\dot{\psi} = \omega(F_1, F_2, x) + \varepsilon \Psi(x, F_1, F_2, y, \dot{y}, \vartheta),$$

$$\dot{\vartheta} = \nu(x) + \varepsilon \Theta(x, y, \dot{y}, \vartheta).$$

Here

$$T = 2\int_{F_2}^{F_1} \frac{dy}{\sqrt{\frac{2}{m}(V(x, F_1) - V(x, y))}}$$

is the period of the oscillations; F_1 and F_2 are related by

$$\int_{F_2}^{F_1} Q(x, y)\, dy = 0;$$

$y = y(F_1, F_2, \psi, x)$, $\dot{y} = \dot{y}(F_1, F_2, \psi, x)$ are determined with the aid of the integrals of the unperturbed system;

$\omega = 2\pi/T$ is the natural frequency in the zeroth approximation; the functions A and Ψ have the form:

$$A^{(i)} = \frac{1}{Q(x, F_1)}\left[\dot{y}^{(i)} f(\dot{y}^{(i)}) - \frac{1}{m}\int_{y}^{F_1} \frac{\partial m Q}{\partial x}\, d\eta\, X(\dot{y}^{(i)})\right],$$

$$\Psi^{(i)} = (-1)^{i+1} 2\pi \left[\frac{\partial}{\partial x}\left(\frac{1}{T}\int_{y}^{F_1} \frac{d\eta}{\sqrt{\frac{2}{m}(V(F_1) - V(\eta))}}\right) X(\dot{y}^{(i)}) + \frac{\partial}{\partial F_1}\left(\frac{1}{T}\int_{y}^{F_1} \frac{d\eta}{\sqrt{\frac{2}{m}(V(F_1) - V(\eta))}}\right) A^{(i)}\right]$$

$$\left(\dot{y}^{(i)} = (-1)^{i}\sqrt{\frac{2}{m}(V(x, F_1) - V(x, y))}\right).$$

(The index $i = 1$ refers to the case when y varies from F_1 to F_2, and the index $i = 2$ to the case when y varies from F_2 to F_1.)

We shall say that resonance occurs in the system, defined by the relatively prime integers p and q, if for certain values of the parameters F_1, F_2, x the equality $p\omega = q\nu$ is satisfied.

Passing in (2) from the variables ϑ, ψ to the variables $\varphi = \vartheta - \frac{p}{q}\psi,\ \beta = \frac{1}{q}\psi$:

$$\dot{F}_1 = \varepsilon A(x, F_1, F_2, y, \dot{y}, \varphi + p\beta),$$

$$\dot{x} = \varepsilon X(x, y, \dot{y}, \varphi + p\beta),$$

$$\dot{\varphi} = \lambda(F_1, F_2, x) + \varepsilon\Phi(x, F_1, F_2, y, \dot{y}, \varphi + p\beta), \tag{3}$$

$$\dot{\beta} = \Omega(F_1, F_2, x) + \varepsilon B(x, F_1, F_2, y, \dot{y}, \varphi + p\beta),$$

where $\lambda = \nu - \frac{p}{q}\omega,\ \Omega = \frac{1}{q}\omega,\ \Phi = \Theta - \frac{p}{q}\Psi,\ B = \frac{1}{q}\Psi$.

Let us introduce, in the following way, the mean values of the functions entering into (3), for fixed values of the parameters F_{10}, F_{20}, x_0:

$$A_1 = \frac{1}{qT_0}\sum_{k=0}^{q-1}\int_{F_{20}}^{F_{10}}\sum_{i=1,\,2}A^{(i)}\left(F_{10}, F_{20}, x_0, y, \dot{y}^{(i)}, \varphi + p\beta\big|_{t=t_0} + \frac{2\pi pk}{q}\right.$$

$$\left.+(-1)^{i+1}\nu_0\int_{y}^{F_{10}}\frac{d\eta}{\sqrt{\frac{2}{m}(V(F_{10}) - V(\eta))}}\right)\frac{dy}{\sqrt{\frac{2}{m}(V(F_{10}) - V(y))}}. \tag{4}$$

Applying to system (3) the scheme developed in (7) for systems of a more general form, we arrive at the following results: the resonant values of the amplitude curves F_{10} and F_{20}, of the parameters x_0, and of the detuning φ_0 are found, in the zeroth approximation with respect to ε, from the equations

$$A_1(x_0, F_{10}, F_{20}, \varphi_0) = 0,$$

$$X_1(x_0, F_{10}, F_{20}, \varphi_0) = 0,$$

$$\lambda(x_0, F_{10}, F_{20}) = 0, \tag{5}$$

$$\int_{F_{20}}^{F_{10}} Q(x_0, y)\, dy = 0.$$

Corrections to the coordinates of the resonance point in the first approximation are found from the linear system

$$\begin{gathered}
\frac{\partial A_1}{\partial F_{10}}\,\delta F_1 + \frac{\partial A_1}{\partial x_0}\,\delta x + \frac{\partial A_1}{\partial \varphi_0}\,\delta\varphi = 0,\\
\frac{\partial X_1}{\partial F_{10}}\,\delta F_1 + \frac{\partial X_1}{\partial x_0}\,\delta x + \frac{\partial X_1}{\partial \varphi_0}\,\delta\varphi = 0,\\
\frac{\partial \lambda}{\partial F_{10}}\,\delta F_1 + \frac{\partial \lambda}{\partial x_0}\,\delta x + \Phi_1(F_{10}, F_{20}, x_0, \varphi_0) = 0.
\end{gathered} \tag{6}$$

Following (^{7}), let us introduce the characteristic equation

$$\det(M - kE) = 0, \tag{7}$$

where E is the identity matrix of the corresponding dimension, and M is a matrix of the form

$$\begin{pmatrix}
\varepsilon\dfrac{\partial A_1}{\partial F_{10}} & \varepsilon\dfrac{\partial A_1}{\partial x_0} & \varepsilon\dfrac{\partial A_1}{\partial \varphi_0}\\
\varepsilon\dfrac{\partial X_1}{\partial F_{10}} & \varepsilon\dfrac{\partial X_1}{\partial x_0} & \varepsilon\dfrac{\partial X_1}{\partial \varphi_0}\\
\dfrac{\partial \lambda}{\partial F_{10}} + \varepsilon\left(\dfrac{\partial \Phi_1}{\partial F_{10}} + \dfrac{\partial^2 \lambda}{\partial F_{10}^2}\delta F_1 + \dfrac{\partial^2 \lambda}{\partial F_{10}\partial x_0}\delta x\right) & \dfrac{\partial \lambda}{\partial x_0} + \varepsilon\left(\dfrac{\partial \Phi_1}{\partial x_0} + \dfrac{\partial^2 \lambda}{\partial x_0^2}\delta x + \dfrac{\partial^2 \lambda}{\partial x_0\partial F_{10}}\delta F_1\right) & \varepsilon\dfrac{\partial \Phi_1}{\partial \varphi_0}
\end{pmatrix}.$$

We shall require that all roots of (7) have negative real parts. In addition, we shall assume that all functions entering into (1) are sufficiently smooth. Under these conditions and under certain additional restrictions on the matrix M (these additional conditions are not written out here), from the results of (^{7}) there follows the following assertion: for arbitrarily large $T > 0$ and arbitrarily small $\xi > 0$ there exists an $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, there exists a $\delta(\varepsilon)$ such that from the condition

$$\max|F_1(t_0) - F_{10},\, x(t_0) - x_0,\, \varphi(t_0) - \varphi_0| < \delta$$

for all $t_0 \le t \le T$ there follows the inequality

$$\max|F_1(t) - F_{10},\, x(t) - x_0,\, \varphi(t) - \varphi_0| < \xi.$$

§ 3. **The case of slow time.** Let the single slowly varying parameter be the "slow time" $\tau = \varepsilon t$. Similar problems were considered in (^{9}). In this case the zero approximations of the coordinates of the stationary point are found from the equations

$$\frac{\partial\lambda/\partial\tau}{\partial\lambda/\partial F_{10}} + A_1(F_{10}, F_{20}, \varphi_0, \tau) = 0,$$

$$\lambda(F_{10}, F_{20}, \tau) = 0, \qquad \int_{F_{20}}^{F_{10}} Q(\tau, y)\, dy = 0.$$

Let us note that the stationary resonance values F_0, φ_0 are functions of τ. The conditions of negativity of the real parts of the roots of (7) take the form

$$\left(\frac{\partial A_1}{\partial F_{10}} + \frac{\partial \Phi_1}{\partial \varphi_0}\right)_{\tau=\tau_0} < 0, \qquad \left(\frac{\partial\lambda}{\partial F_{10}}\frac{\partial A_1}{\partial \varphi_0}\right)_{\tau=\tau_0} < 0.$$

The principal assertion of § 2 is also valid for this case.

§ 4. **Physical examples.** As a first example, let us consider the generalized equation of motion of electrons in the Barkhausen–Kurz oscillator (see ([6])). The equation of motion has the form

$$\ddot{y} + \operatorname{sign} y = \varepsilon(a\cos\nu t - b\dot{y}).$$

In this case the oscillations occur within symmetric limits: $F_2 = -F_1$. Consider the resonance regime determined by the numbers $p = 2n+1$, $q = 1$ (n a natural number). The stationary values of the amplitude and detuning have the form

$$F_{10} = \frac{\pi^2 p^2}{8\nu^2}, \qquad \sin(\varphi_0 + p\beta_0) = \frac{(-1)^n \pi^3 p^3 b}{24 a\nu}.$$

The stability conditions are written in the form

$$\left\{(-1)^n a\cos(\varphi_0 + p\beta_0) < 0, \qquad \frac{b}{a}\left[(-1)^n\left(\frac{\pi p}{4} - 1\right) - \frac{3}{2}\right] < 0.\right.$$

As a second example, consider the mechanical oscillations of a body situated between two initially strongly compressed springs. (For details concerning the construction of such a system, see ([4],[5]), where a similar problem was considered for the nonresonant case.) To study resonant regimes we take the equation of motion in the form

$$\frac{d}{dt}[m\dot{y}] + Q(\tau, y) = \varepsilon(a\sin\nu t - \beta\dot{y}).$$

Here $Q = a > 0$ for $y > 0$, and $Q = -b$ ($b > 0$) for $y < 0$; the parameters $a, b, \nu, m, \alpha, \beta$ are functions of the slow time $\tau = \varepsilon t$. In this case the oscillations are no longer symmetric: $F_2 = -\dfrac{a}{b}F_1$.

For a sufficiently small coefficient of friction β and sufficiently slowly varying coefficients a, b, m, ν, for each value of p (we take q equal to 1) there exist two stationary resonant regimes, determined by the expressions

$$F_{10} = \frac{a\pi^2 p^2 b^2}{2m\nu^2(a+b)^2}, \qquad \cos(\varphi_0 + p\beta_0) = \frac{a^2\pi^3 p^3 b^2}{\alpha\nu(a+b)^3 \sin\dfrac{\pi p b}{a+b}} \times$$

$$\times \left\{ \left(\frac{b}{a+b}\sqrt{\frac{a}{m}}\right)^{-1} \left(\frac{b}{a+b}\sqrt{\frac{a}{m}}\right)' + \frac{\nu'}{\nu} - \frac{(3a+b)(am)'}{12am(a+b)} + \frac{a(bm)'}{6bm(a+b)} - \frac{\beta}{3m} \right\}.$$

The stability conditions for these regimes are written in the form

$$\alpha \sin\frac{\pi p b}{a+b} \sin(\varphi_0 + p\beta_0) < 0,$$

$$\frac{\alpha\nu}{2\pi^2 p^2 a^2}\left[(a+b)(-1)^p + \frac{(a+b)(a+5b)}{\pi p b}\sin\frac{\pi p b}{a+b} + (3a-b)\cos\frac{\pi p b}{a+b}\right] \times$$

$$\times \cos(\varphi_0 + p\beta_0) + \sqrt{\frac{m}{a}}\left(\frac{b}{a+b}\sqrt{\frac{a}{m}}\right)' - \frac{b\nu'}{\nu(a+b)} < 0.$$

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.