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Abstract

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MATHEMATICAL PHYSICS

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A NEW VARIATIONAL FORMALISM IN GENERAL RELATIVITY

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The equations of the gravitational field have the form (1)

$$R_i^k - {}^1/_2 \delta_i^k R = \varkappa T_i^k. \tag{1}$$

When the distance l is changed at constant velocity (with $\varkappa = \text{const}$), the left-hand side of equations (1) changes as l^{-2}, while the right-hand side changes as l^{-3}.

Consequently, the left- and right-hand sides are transformed according to different transformation groups, which violates the law of conservation of energy and makes equations (1) unsatisfactory. When the scale is changed, all terms of equations (1), on the contrary, are transformed according to one group. It is fairly obvious that it is immaterial whether one changes the scale or the distance. Let us put forward the assumption that equations expressing world laws must be invariant with respect to the application of each of these operations, as, for example, Maxwell' s equations are. Since the equations of the gravitational field (1) depend on what we change, one may conclude that from this point of view they are incorrect. Therefore one should adopt the hypothesis that, if it is necessary to vary different Lagrangians, then they must be chosen so that all the terms obtained as a result of varying the equations are transformed according to one and the same transformation group, i.e. in deriving these equations one must consistently apply the principles set forth in Noether' s theorem.

The gravitational constant $\varkappa$ may be represented in the form

$$\varkappa = l/E, \tag{2}$$

where E is a certain energy. If energy is to be conserved when the volume of the Universe is changed, then this, generally speaking, is not true for any previously chosen characteristic length on which $\varkappa$ may depend. Therefore there are no grounds for considering that the dimensional quantity $\varkappa$ cannot vary with time and depend on local conditions in the Universe, i.e. that $\varkappa$ may depend on the coordinates x^i. The field equations and the conservation of energy-momentum are obtained by varying some scalar quantity characterizing this field; moreover, in the case of the gravitational field such a scalar quantity need not necessarily be R, but, generally speaking, may be any function of R.

It is further obvious that, when varying different scalars characterizing different fields, the scalars must be chosen so that the connection between them is established only with the aid of quantities that are truly conserved dimensionally, such as energy, momentum and angular momentum or energy, the speed of light, and the characteristic length l_0 characterizing the dimensions of elementary particles. It should be added to this that, of course, in each Lagrangian the characteristic dimensional quantities must enter in such a way that the Lagrangian has the dimension of energy density, i.e. pressure.

In the case of the gravitational field, in equation (2) E may be understood as the quantity $\bar{E}_0$, the total energy of the given model of the "Universe," and l as the quantity a, where a is the radius of curvature characterizing the dimensions of the "Universe." Indeed:

$$\varkappa = 8\pi G/c^4 = {}^8/_3\pi \cdot 10^{-7}/0.81 \cdot 10^{42} = 2 \cdot 10^{-48} \text{ cm/erg}.$$

Since $l = a \simeq 10^{28}$ cm, $\bar{E}_0 = 10^{76}$ erg ($\bar{E}_0 = M_0c^2 = Nm_pc^2 \simeq 10^{76}$ erg, M_0 is the mass of the metagalaxy, N is the number of nucleons, m_p is the mass of a nucleon)

then $a/\bar{E}_0 = 10^{29}/10^{76} = 10^{-48}$ cm/erg $\simeq 8\pi G/c^4 = \varkappa$. The quantity a is variable in dynamical models; therefore the quantity $\varkappa$ must also vary, since $\bar{E}_0 = \text{const}$, namely $\varkappa \sim a$.

If we choose the Lagrangian of the gravitational field in the form

$$L_g = \frac{1}{3} E_0 R_0^{3/2}, \tag{3}$$

where R_0 is the scalar curvature of the "Universe" (metagalaxy), $E_0 = \text{const} \cdot \bar{E}_0$, then, since $\bar{E}_0 = \text{const}$, we arrive at the fact that the Lagrangian (3) is transformed under a change of l in the same way as the Lagrangian of matter, and the same will hold under a change of scale. Consequently, the equations that we obtain as a result of variation will be transformed in both cases according to one group, which is necessary for the strict fulfillment of the laws of conservation of energy-momentum.

Let us proceed to the derivation of the field equation and the conservation laws. Using directly the variational equations of Lagrange for this purpose, varying either only L_g (assuming $\delta\Gamma^l_{ik} \neq 0$) or L_g and L_m [2,3], we arrive at the equations

$$\frac{3}{2}R_{0ik} - \frac{1}{2}g_{0ik}R_0 = \frac{3}{2E_0\sqrt{R_0}}T_{0ik} = \varkappa T_{0ik}, \tag{4}$$

where $\varkappa = 3/2E_0\sqrt{R_0}$; T_{0ik} is the matter tensor. The pseudotensor

$$t^k_{0i} = -\delta^k_i \frac{R_0}{2\varkappa} + \frac{3}{2}\bar{t}^k_i + \frac{E_0}{2}\frac{\partial\sqrt{R_0}}{\partial x^r}\frac{\partial g^{ml}}{\partial x^i}\left(g^{kr}g_{ml} - \delta^k_m\delta^r_l\right), \tag{5}$$

where $2\varkappa\bar{t}^k_i = g^{km}\partial\Gamma^l_{ml}/\partial x^i - g^{ml}\partial\Gamma^k_{ml}/\partial x^i$, and the conservation equations hold:

$$\frac{\partial}{\partial x^k}\left[\sqrt{-g}\left(T^k_{0i} + t^k_{0i}\right)\right] = \sqrt{-g}\left[\frac{1}{2}\frac{\partial g^{ml}}{\partial x^i}\theta_{ml} + E_0\frac{\partial\sqrt{R_0}}{\partial x^k}R^k_{0i}\right], \tag{6}$$

where

$$\frac{\theta_{ik}}{E_0} = \frac{\partial\sqrt{R_0}}{\partial x^l}\left(g_{im}g^{lr}\Gamma^m_{kr} - g_{ik}g^{mr}\Gamma^l_{mr}\right) + \frac{\partial^2\sqrt{R_0}}{\partial x^l\partial x^m}\left(g^{lm}g_{ik} - \delta^m_l\delta^l_k\right). \tag{7}$$

Since $\bar{T}^k_{i;k} = 0$, from this and from the field equations and the Bianchi identities we have

$$\frac{\partial\sqrt{R_0}}{\partial x^k}R^k_{0i} = 0.$$

These equations, for example, are identically satisfied for a homogeneous and isotropic Universe, when $R^0_{00} = 0$ ([4]).

Equations (4) describe the behavior of matter in its own gravitational field, in any case for the metagalaxy. The conservation equations for momentum–energy (6) do indeed describe the conservation laws in the metagalaxy, since the original equations possess an external conformal group.

In the Universe (in the metagalaxy), individual significant inhomogeneities are observed: stars, their clusters, galaxies. In our view, the old Einstein equations with $\varkappa = \text{const}$ do not describe the development of matter in the metagalaxy, but they do describe rather well the local inhomogeneities in it, when the variation of $\varkappa$ can be neglected. One may think that the Einstein field equations (1) and our equations (4) are equations describing two opposite limiting cases: relatively small condensations of matter over periods of time small in comparison with the age of metagalaxies (Einstein equations (1)) and a comparatively homogeneous metagalaxy over the entire time of its existence (equation (4)).

There arises the necessity of deriving general equations of the gravitational field that describe both the local inhomogeneities in the metagalaxy and its common field. For this purpose one should use the Lagrangian

$$L = \frac{E_0\sqrt{R_0}}{3}R = \frac{R}{2\chi}, \tag{8}$$

where χ is a variable quantity depending on the energy and dimensions of the metagalaxy.

Application of the general variational equations leads to the equations

$$\left(R_{ik} - \frac{1}{2}g_{ik}R - \chi T_{ik}\right)\delta g^{ik} + \frac{R}{2R_0}R_{0ik}\delta g_0^{ik} = 0, \tag{9}$$

where δg_0^{ik} refers to the field of the metagalaxy, while δg^{ik} refers to the common field, characterized by the quantities g^{ik}, with $\delta g^{ik} = \delta g^{ik} + \delta g_0^{ik}$, where g^{ik} are the components of the metric tensor characterizing only the additional local field.

One may write that $\delta g_0^{ik} = A_{ab}^{ik}\delta g^{ab}$, where A_{ab}^{ik} are the components of a dimensionless tensor which, at each point of 4-space, relates the components of the variations of the metric tensors g_0^{ik} and g^{ik}. Since the variations δg^{ik} are arbitrary, (9) assumes the form

$$R_i^k - \frac{1}{2}\delta_i^k R + \frac{R}{2R_0}R_{0a}^b A_{ib}^{ka} = \chi T_i^k. \tag{10}$$

If $\delta g^{ik} = 0$, then $R \to R_0$ and (10) becomes (4). If $\delta g^{ik} \gg \delta g_0^{ik}$, $A_{ab}^{ik} = 0$, then (10) becomes the ordinary Einstein equations (1), which are valid in the neighborhoods of large inhomogeneities of matter with a density greatly increased in comparison with the mean value.

It is convenient to write equation (10) in another form. Let us use equation (4) and eliminate from (10) $R_{0a}^b = {}^2/_3\chi T_{0a}^b + {}^1/_3\delta_a^b R_0$; then (10) will take the form

$$R_i^k - \frac{R}{2}\left(\delta_i^k - \frac{A_i^k}{3}\right) = \chi\left(T_i^k - \frac{R}{3R_0}A_{ib}^{ka}T_{0a}^b\right). \tag{11}$$

As an approximate but interesting example, let us derive interpolation equations for the case of an "internal" or external Schwarzschild field placed in the general gravitational background of the metagalaxy. Choosing some point at the center of the local field, created, for example, by the Sun, we find that in this case, when the proper time for radial motions for the Friedmann field is: $g^{00} = -1$, with $g_0^{11} = 1/(1-r^2/a^2) = 1/[1-(r_g/r)(\bar\rho/\rho)]$; $\bar\rho = 3M/4\pi r^3$, $r_g = 2GM/c^2 = 8\pi G r^3\bar\rho/c^2$, $\bar\rho$ is the mean density at distance r; ρ_0 is the mean density of the metagalaxy, with $8\pi G\rho_0 a^2/3c^2 = 1$. For the "Schwarzschild" metric $g^{00} = -1$; $g^{11} = 1/[1-(r_g/r)]$ (the other g^{ik} are identical for both fields). Here by a

one must understand the "radius" of the metagalaxy. It is evident that $\delta g^{11} = \delta g_0^{11}(1+\bar{\rho}/\rho_0) = \delta g_0^{11}\rho\ /\rho_0$, where $\rho\ = \rho_0 + \bar{\rho}$ is the total density, $A = \rho_0/\rho$.

For $R \gg R_0$ we have equation (1),

For $R = R_0$ we have equation (4).

The laws of conservation of energy for the Lagrangian $L = R/2\chi$ at $\chi = \text{const}$ do not apply, since in regions of local inhomogeneities the energy of these regions interacts with the energy of the metagalaxy, and the conservation laws are fulfilled only for the metagalaxy at $R = R_0$, $\chi \sim a$. The quantities $g^{11} = g_0^{11}$ when $r_0/r = r^2/a^2$, which gives

$$r^3 = r_g a^2 = 2GMa^2/c^2 = \text{const}\cdot 2GM\tau^2 = 10^{28}M, \tag{12}$$

where const $\simeq 1$. For $r_g \ll a$ Newton' s law holds; for $r_g \to a$, $r = a \simeq c\tau$, which holds for the entire metagalaxy. It is evident that, for

For $r^3 < r_g a^2$ the Newton–Einstein laws act; for $r^3 > r_g a^2$ the law of the general expansion of the metagalaxy comes into force.

It is interesting to note that for nucleons (particles) in the Universe $r^3 = 10^4$ cm^3, $r = 20$ cm, which corresponds to the mean density of nucleons in the metagalaxy ($\rho_0 = 10^{-28}$ g/cm^3). For stars $r^3 = 10^{60}$ cm^3, $r = 10^{20}$ cm, which corresponds to their mean distance at the edge of the galaxy; for galaxies $r^3 = 10^{72}$ cm^3, $r = 10^{24}$ cm, which also corresponds to the mean distance between galaxies. For the metagalaxy $r^3 = a^3 = 10^{84}$ cm; $r = a = 10^{28}$ cm, which also corresponds to reality.

The equality $r_{0g} = a = r$ is a consequence of the known expression: $2GM_0/c^2 = r_{0g} = a = 8\pi\rho_0 G r^3/c^2$. Under the assumption that $G \sim a \simeq ct$, we have $r^3 \sim \tau$, since $M\tau^2 = \text{const}$. For $G = \text{const}$, $r^3 \sim \tau^2$. It is difficult to suppose that we live precisely in that "fortunate" epoch for physicists when both these laws give coincident quantities, the gravitational radius of the metagalaxy is equal to its ordinary radius (the radius of curvature), and χ is expressed precisely as the ratio of the radius of the metagalaxy to its energy.

Since the changes with time of the stable distances between objects of different classes in the metagalaxy (nucleons, stars, and galaxies) are different under the two limiting hypotheses ($G \sim a$ and $G = \text{const}$), there is hope of finding out from astronomical data which of these two laws corresponds to reality. For $G = \text{const}$, objects must formerly have been located closer together than under the law $G \sim a$. For example, 5 billion years ago the volume per nucleon, star, or galaxy for $G = \text{const}$ was 2 times smaller than for $G \sim a$, and amounted to approximately 1/4 of the mean present-day volume.

For nucleons and other particles in a space free of local inhomogeneities, in addition to the Newtonian forces of attraction, there acts the pressure of the general gravitational field $p_g = GM_0^2/a^4 \simeq Gm_p^2/r^4 \simeq 10^{-7}$ dyn/cm^2, where

$r_p \simeq 10^{-13}$ cm, which also leads to an apparent violation of Newton' s law of attraction. The same situation occurs at large distances from local inhomogeneities.

One may think that the stable mean distances are precisely the distances $r = (r_g a^2)^{1/3}$, since an increase in distance will be hindered by the "equilibrium" metagalactic background of gravitation. In the case $G \sim a$, when $r^3 = r_g a^2$, the law of motion changes. If for $r^3 < r_g a^2$ the acceleration $g \neq 0$ (the force of attraction acts), then for $r^3 \gg r_g a^2$ particles move by inertia with the general background of the metagalaxy, which leads to definite "stable" distances. For $G = \text{const}$, the law of motion is approximately the same everywhere and the acceleration is everywhere nonzero; therefore "stable" distances can hardly exist.

Since for particles of four classes (nucleons, stars, galaxies, metagalaxies) there exist definite "stable" distances at each given moment of time, this speaks in favor of the hypothesis $G \sim a$.

Since the sizes of galaxies, according to the conclusions of astronomers, can increase with time only very insignificantly, this too speaks in favor of our hypothesis $G \sim a$.

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CITED LITERATURE

1. L. D. Landau, E. M. Lifshitz, *Field Theory*, 3rd ed., Moscow, 1960, § 95.
2. K. P. Stanyukovich, DAN, 154, No. 2 (1964).
3. K. P. Stanyukovich, DAN, 153, No. 3 (1963).
4. K. P. Stanyukovich, DAN, 151, No. 3 (1963).

Note: Figure translations are in progress. See original paper for figures.