# MATHEMATICS

I. V. VITENKO, A. N. KOSTOVSKII

1964

SovietRxiv

## Abstract

## Full Text

MATHEMATICS

I. V. VITENKO, A. N. KOSTOVSKII

# DETERMINATION OF THE PRINCIPAL INDICES OF LAURENT SERIES

*(Presented by Academician A. A. Dorodnitsyn, 20 XII 1963)*

Let a Laurent series be given

$$f(z) = \sum_{m=-\infty}^{\infty} a_m z^m, \tag{1}$$

converging in the annulus $r < |z| < R$.

If $f(xe^{i\varphi}) \neq 0$ for $0 \leq \varphi < 2\pi,\ r < x < R$, then the value

$$I(x) = \frac{1}{2\pi i} \int_{|z|=x} \frac{f'(z)}{f(z)}\, dz \tag{2}$$

is called the **principal index** of the series (1).

In ([1]) sufficient conditions are given for the existence of the principal index for values of $x$ from the interval $(r, R)$. In ([2]) necessary and sufficient conditions for the principal index are given when $f(z)$ is a polynomial. In the present note the results of ([2]) are generalized to Laurent series, and also a number of other results are given.

Introduce the notation

$$u_m^{(k)} = \min_{\alpha>0} \left| a_m^{(k)} : a_{m+\alpha}^{(k)} \right|^{1/\alpha}, \qquad v_m^{(k)} = \max_{\beta>0} \left| a_{m-\beta}^{(k)} : a_m^{(k)} \right|^{1/\beta},$$

$$D_m^{(k)} = u_m^{(k)} : v_m^{(k)}, \tag{3}$$

where $a_{m-\beta}^{(k)},\ a_m^{(k)},\ a_{m+\alpha}^{(k)}$ are the coefficients of the transformed Laurent series

$$f_k(z) = \sum_{m=-\infty}^{\infty} a_m^{(k)} z^m = \prod_{j=0}^{k-1} f\left(\omega_j^{(k)} z^{1/k}\right), \qquad \omega_j^{(k)} = \exp\left(\frac{2\pi i}{k} j\right),$$

$$r^k < |z| < R^k \qquad (k = 1, 2, \ldots).$$

Choose $x_1$ and $x_2$ so that $f(z) \neq 0$ for $x_1 < |z| < x$ and $x < |z| < x_2$, and $r \leq x_1 < x < x_2 \leq R$.

Using the relations between the quantities $a_m^{(k)}$ and $x$ given in (3), we can write:

$$\lim_{k\to\infty} \frac{u_p^{(k)}}{x^k} > 0, \qquad \frac{v_p^{(k)}}{x^k} = O(\rho^k) \qquad \left(\frac{x_1}{x} < \rho < 1\right),$$

$$\lim_{k\to\infty} \frac{u_{p+j}^{(k)}}{x^k} > 0, \qquad \lim_{k\to\infty} \frac{v_{p+j}^{(k)}}{x^k} < \infty \qquad (j = 1, 2, \ldots, q-1), \tag{4}$$

$$\frac{x^k}{u_{p+q}^{(k)}} = O(\tau^k), \qquad \lim_{k\to\infty} \frac{v_{p+q}^{(k)}}{x^k} < \infty \qquad \left(\frac{x}{x_2} < \tau < 1\right),$$

$$p = I(x - 0), \qquad q = I(x + 0).$$

From (3) and (4) it follows

**Theorem 1.** *In order that the number p from (2), which satisfies the inequalities* $I(r+0) \leq p \leq I(R-0)$, *be a principal index, it is necessary and sufficient that*

$$\lim_{k\to\infty} D_p^{(k)} = \infty. \tag{5}$$

Without the requirement $I(r+0) \leq p \leq I(R-0)$, condition (5) of the theorem will be necessary but, generally speaking, will not be sufficient. This is easy to see from the following example:

$$f(z) = \sum_{m=0}^{\infty} a_m z^m = (z-2) \exp\left(\sum_{m=0}^{\infty} \frac{1}{r_m^2} z^{r_m}\right),$$

where $r_0, r_1, \ldots$ is an increasing sequence of prime numbers of the natural series. This series has a unique principal index, equal to zero; nevertheless $\lim_{k\to\infty} D_1^{(k)} = \infty$, $k = h, h^2, \ldots, h^\nu, \ldots$; $h$ is an integer $\geq 2$.

**Theorem 2.** *If the series (1) converges in the domain* $0 < |z| < \infty$, *then condition (5) is necessary and sufficient for the number p from (2) to be a principal index.*

In ([1]) it is shown that if $D_p^{(1)} > 9$ for the function (1), then for this function the number $p$ will be a principal index. Therefore condition (5) in Theorems 1 and 2 may be replaced by the relation

$$\max_{k\geq 1} D_p^{(k)} > 9,$$

where $D_p^{(k)}$ in (3) is taken for $f_k(z)$.

In particular, if (1) is a Taylor series ($a_m = 0$ for $m < 0$), having in its disk of convergence of radius $R$ the zeros $0 \leq |z_1| \leq |z_2| \leq \cdots \leq |z_n| \leq \cdots < R$, then Theorem 2 may be formulated as follows:

**Theorem 2′.** *For the inequality $|z_n| < |z_{n+1}|$ (or $|z_n| < R$, if the power series has only $n$ zeros in the disk of convergence) to hold between the moduli of the zeros of a Taylor series, it is necessary and sufficient that*

$$\lim_{k\to\infty}\left(\left|\frac{a_{n-\beta}^{(k)}}{a_n^{(k)}}\right|^{1/\beta} : \min_{\alpha>0}\left|\frac{a_n^{(k)}}{a_{n+\alpha}^{(k)}}\right|^{1/\alpha}\right) = 0 \qquad (\beta = 1, 2, \dots, n).$$

Let an arbitrary function $v(t)$, holomorphic in the annulus $r < |z| < R$, be given. Knowing the principal indices of the series (1), we compute the value of the function

$$I_v(x) = \frac{1}{2\pi i}\int_{|z|=x} \frac{f'(z)}{f(z)} v(z)\, dz, \qquad r < x < R.$$

Consider the auxiliary Laurent series

$$Q_{k,v}(z) = \frac{1}{k}\sum_{l=0}^{k-1}\prod_{\substack{j=0\\ j\neq l}}^{k-1} f\left(\omega_j^{(k)} z^{1/k}\right) F_v\left(\omega_l^{(k)} z^{1/k}\right) = \sum_{m=-\infty}^{\infty} b_{m,v}^{(k)} z^m, \tag{6}$$

where

$$F_v(u) = u f'(u) v(u), \qquad r < |u| < R, \qquad r^k < |z| < R^k.$$

Suppose that the function $f(z)$ has no zeros in the annulus $r \leq x_1 < |z| < x_2 \leq R$; then in the annulus $x_1^k < |z| < x_2^k$ the function $Q_{k,v}(z)$ can be represented in the form

$$Q_{k,v}(z) = f_k(z)\frac{1}{k}\sum_{l=0}^{k-1} T\left(\omega_l^{(k)} z^{1/k}\right) = f_k(z)\sum_{m=-\infty}^{\infty} t_m z^m, \tag{7}$$

where

$$T(z) = z\frac{f'(z)}{f(z)}\, v(z) = \sum_{m=-\infty}^{\infty} t_m z^m, \qquad t_0 = I_v(x)$$

$$(x_1 < |z| < x_2; \quad x_1 < x < x_2).$$

Using the relations for the quantities $a_m^{(k)}$ and $t_m$ given in $(3, 4)$, from (6) and (7) we can write

$$I_v(x) = -\frac{b_{p,v}^{(k)}}{a_p^{(k)}} + O(\rho^k) \qquad \left(\frac{x_1}{x_2} < \rho < 1\right), \tag{8}$$

$$p(x) = I(x) \qquad (x_1 < x < x_2).$$

Suppose now that in the annulus $r \leq r' < |z| < R' \leq R$ the function $f(z)$ has in all $q$ zeros $z_1, z_2, \ldots, z_q$; then

$$I_v(R' - 0) - I_v(r' + 0) = \sum_{m=1}^{q} v(z_m).$$

Hence, from (8), we obtain the theorem.

**Theorem 3.** If the Laurent series (1) in the annulus $r' < |z| < R'$ has in all $q$ zeros $z_1, z_2, \ldots, z_q$, then

$$\lim_{k\to\infty}\left(\frac{b_{p+q,v}^{(k)}}{a_{p+q}^{(k)}} - \frac{b_{p,v}^{(k)}}{a_p^{(k)}}\right) = \sum_{m=1}^{q} v(z_m), \qquad p = I(r' + 0), \tag{9}$$

where $v(z)$ is an arbitrary holomorphic function in this annulus.

The coefficients $a_m^{(k)}, b_m^{(k)}$ are determined by the formulas proposed in (4). In particular, if $k$ takes the discrete values $1, 2, 4, 8, 16, \ldots$, then

$$a_m^{(2k)} = (-1)^m \left[(a_m^{(k)})^2 + 2\sum_{j=1}^{\infty}(-1)^j a_{m-j}^{(k)} a_{m+j}^{(k)}\right], \tag{10}$$

$$b_m^{(2k)} = (-1)^m \left[a_m^{(k)} b_{m,v}^{(k)} + \sum_{j=1}^{\infty}(-1)^j \left(a_{m-j}^{(k)} b_{m+j,v}^{(k)} + a_{m+j}^{(k)} b_{m-j,v}^{(k)}\right)\right],$$

$$m = 0, \pm 1, \pm 2, \ldots$$

Here

$$a_m^{(1)} = a_m, \qquad \sum_{m=-\infty}^{\infty} b_{m,v}^{(1)} z^m = zf'(z)v(z), \qquad r < |z| < R.$$

Taking $v(z)$ successively equal to $z, z^2, \dots, z^q$, we find the coefficients of the functions $f_k(z), Q_{k,z}(z), Q_{k,z^2}(z), \dots, Q_{k,z^q}(z)$. The transformation of the coefficients of these functions is carried out by formulas (10) or by the formulas given in (4). By formulas (9) we find $\sum_{m=1}^{q} z_m, \sum_{m=1}^{q} z_m^2, \dots, \sum_{m=1}^{q} z_m^q$, and hence we can compose a polynomial whose roots will be the roots $z_1, z_2, \dots, z_q$ of the function (1) from the annulus $r' < |z| < R'$.

Lviv State University
named after Ivan Franko

Received
30 XI 1963

## REFERENCES

1. A. Ostrowskie, Acta Math., **72**, 99 (1940).
2. A. N. Kostovskii, DAN, **147**, No. 2 (1962).
3. I. V. Vitenko, Dokl. USSR, **9**, No. 1 (1963).
4. I. V. Vitenko, A. N. Kostovskii, *Theoretical and Applied Mathematics*, Lviv, issue II, 31 (1963).

*Note: Figure translations are in progress. See original paper for figures.*