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Abstract

Full Text

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SINGULAR INTEGRAL EQUATIONS ON AN INFINITE CYLINDER

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In the present work a theory is constructed for singular integral equations on an infinite m-dimensional cylinder. Operators generated by periodic singular kernels are considered.

1. Let Ω be a strip of m-dimensional Euclidean space E_m,

$$\Omega = \{\, |x^{(i)}| \leq 1/2,\ i = 1, \dots, k;\ -\infty < x^{(j)} < +\infty,\ j = k+1, \dots, m;\ k < m \,\}.$$

This strip will subsequently play the role of the development of an m-dimensional cylinder. Put

$$x_1 = (x^{(1)}, \dots, x^{(k)}, 0, \dots, 0), \quad x_2 = (0, \dots, 0, x^{(k+1)}, \dots, x^{(m)}),$$

$$n = (n^{(1)}, \dots, n^{(k)}, 0, \dots, 0),$$

where all $n^{(i)}$ are integers. By xy we shall denote the scalar product of the vectors x and y. The Euclidean $(m-k)$-dimensional space of the variables $x^{(k+1)}, \dots, x^{(m)}$ will be denoted by Ω_2, and the k-dimensional cube $|x^{(i)}| \leq 1/2$ $(i = 1, \dots, k)$ by Ω_1. Let also

$$dx_1 = dx^{(1)} \cdots dx^{(k)}, \qquad dx_2 = dx^{(k+1)} \cdots dx^{(m)}.$$

Consider the integrals

$$(2\pi)^{\frac{k-m}{2}} \int_{\Omega_1} dy_1 \left(\lim_{A\to\infty} \int_{|y_2|<A} f(y_1, y_2) \exp(-2\pi i n y_1 - i x_2 y_2)\, dy_2 \right); \quad (1)$$

$$(2\pi)^{\frac{k-m}{2}} \lim_{A\to\infty} \int_{|z_2|<A} dy_2 \int_{\Omega_2} f(y_1, y_2) \exp(-2\pi i n y_1 - i x_2 y_2)\, dy_1. \quad (2)$$

If $f(y_1, y_2) \in L_2(\Omega)$, then the integrals (1) and (2) exist for almost all $x_2 \in \Omega_2$ and coincide almost everywhere. In what follows, repeated integrals of this kind will be denoted as double integrals.

Introduce the space $\widetilde{L}_2(\Omega_2)$ of vector-functions

$$\tilde{f} = \{\tilde{f}_n(x_2)\}$$

with norm

$$\|\tilde{f}\|^2 = \sum_n \int_{\Omega_2} |\tilde{f}_n(x_2)|^2 \, dx_2.$$

In the summation, each coordinate of the vector n ranges over all integers from $-\infty$ to $+\infty$. Define the Fourier transform in the strip Ω by the formula

$$Ff = \left\{ \tilde{f}_n(x_2) = (2\pi)^{\frac{k-m}{2}} \int_\Omega f(y_1, y_2) \exp(-2\pi i n y_1 - i x_2 y_2) \, dy \right\}.$$

Lemma 1. A. The mapping

$$F : L_2(\Omega) \to \widetilde{L}_2(\Omega_2)$$

is isometric.

B. *(inversion formula).*

$$f(x_1, x_2) = 2\pi^{\frac{k-m}{2}} \sum_n \int_{\Omega_2} \tilde{f}_n(y_2) \exp(2\pi i n x_1 + i x_2 y_2) \, dy_2,$$

where the series on the right converges in $L_2(\Omega)$.

C. *(convolution rule).* Let the functions $f(x_1, x_2)$ and $g(x_1, x_2)$ be periodic in the variables $x^{(1)}, \dots, x^{(k)}$, with period equal to one. Then

$$F(f * g) = \{(2\pi)^{\frac{m-k}{2}} \tilde{f}_n(x_2) \tilde{g}_n(x_2)\}.$$

2. We proceed to the construction of a periodic singular kernel. The idea of such a construction is due to Calderón and Zygmund ([1]).

Let

$$K(x_1 - y_1, x_2 - y_2) = \frac{f(\theta)}{r^m}, \qquad r = |x - y|, \qquad \theta = \frac{y - x}{r}.$$

As usual, we assume that

$$\int_{S^{m-1}} f(\theta) \, dS = 0,$$

where S^{m-1} is the unit $(m-1)$-dimensional sphere. Put

$$L(x_1, x_2) = \sum_n K(x_1 - n, x_2).$$

We shall call $L(x_1, x_2)$ a **periodic singular kernel**. Let

$$K_{\varepsilon,N}(x_1, x_2) = \begin{cases} K(x_1, x_2), & \text{if } |x| \geq \varepsilon \text{ and } |x_2| \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

and put

$$L_{\varepsilon,N}(x_1, x_2) = K_{\varepsilon,N}(x_1, x_2) + \sum_n{}' K_{0,N}(x_1 - n, x_2).$$

Theorem 1. Let $|f(\theta)| \leq c$. For all x_2, n such that $|x_2| + |n| > 0$, there exists the limit

$$\Phi_n(x_2) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\Omega L_{\varepsilon,N}(y_1, y_2) \exp(-2\pi i n y_1 - i x_2 y_2)\, dy_1 dy_2 =$$

$$= \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon < |y| < N} K(y_1, y_2) \exp(-2\pi i n y_1 - i x_2 y_2)\, dy_1 dy_2.$$

The functions $\Phi_n(x_2)$ are uniformly bounded with respect to n and x_2.

3. It is known ([2], [3]) that the limit

$$\Phi(x_1, x_2) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon < |y| < N} K(y_1, y_2) \exp(-i x_1 y_1 - i x_2 y_2)\, dy_1 dy_2$$

is a homogeneous function of order zero.

Theorem 1 gives

$$\Phi_n(x_2) = \Phi(2\pi n, x_2). \tag{3}$$

Now let $\Phi(x_1, x_2)$ be an arbitrary homogeneous function of order zero, and let the sequence $\{\Phi_n(x_2)\}$ be defined by formula (3). Let us also assume that

$$u_n(x_2) = \int_{\Omega_1} u(x_1, x_2) \exp(-2\pi i n x_1)\, dx_1$$

and denote by F the Fourier transform on the hyperplane Ω_2.

Definition 1. By a singular integral operator on the m-dimensional cylinder with symbol $\Phi(x_1, x_2) = \Phi(\theta)$ $(\theta = x/|x|)$, we shall mean an operator of the form

$$Au = \sum_n \exp(2\pi i n x_1)\, F^{-1}\Phi_n F u_n. \tag{4}$$

Theorem 2. Let the symbol of the singular operator (4) have continuous derivatives with respect to the Cartesian coordinates of the point θ up to order m inclusive. Then the operator is bounded in $L_p(\Omega)$, $1 < p < \infty$.

For the case $p = 2$ the theorem is obtained directly from the definition of the operator and the properties of the Fourier transform F, and the norm of the operator is equal to $\operatorname{vrai\,max}|\Phi(\theta)|$. In the general case one uses Marcinkiewicz's theorem on multipliers of Fourier series (cf. (2), the theorem on a multiplier of the Fourier integral).

We now introduce operators with symbols depending on the pole. Let $\Phi(x, \theta)$ $(x \in \Omega,\ \theta \in S^{m-1})$ be a function periodic (with period equal to one) in the coordinates $x^{(1)}, \dots, x^{(k)}$ of the point x, and let $\Phi(x,\theta) \in \widehat{W}_2^l(S^{m-1})$ ([2]), where l is an integer and $l \geq 5m/2 - 1$; let $\{Y_{km}(\theta)\}$ be m-dimensional spherical functions forming a complete system in $L_2(S^{m-1})$. By A_{km} we denote the singular operator with symbol $Y_{km}(\theta)$, and by T a completely continuous operator in $L_p(\Omega)$. To a completely continuous operator we assign the symbol identically equal to zero.

Let the function $\Phi(x, \theta)$ be expanded in the series

$$\Phi(x,\theta) = \sum_{k=0}^{\infty} a_{km}(x) Y_{km}(\theta).$$

From the smoothness assumptions on $\Phi(x, \theta)$ it follows that the series $\sum_{k=0}^{\infty} a_{km}(x) A_{km}$ converges in the operator norm.

Definition 2. An operator of the form

$$Au = \sum_{k=0}^{\infty} a_{km}(x) A_{km} u + Tu$$

is called a **singular integral operator with symbol** $\Phi(x, \theta)$.

4. The fundamental point in the theory of singular equations is the rule of multiplication of symbols: the product of singular operators is a singular operator whose symbol is equal to the product of the symbols of the factors ([2]). For operators whose symbols do not depend on the pole this rule is obvious. In the general case the proof is based on the following lemma.

Lemma 2. Let $K(x-y) = Y_{km}(\theta)/r^m$. The functions

1) $\sum_n'\{K(x_1+n,x_2)-K(n,x_2)\}$,

2) $\sum_n' K(n,x_2)$

belong to the space $L_2(\Omega)$.

The convergence of the first series and the belonging of its sum to $L_2(\Omega)$ follow from the estimate

$$|K(x_1+n,x_2)-K(n,x_2)| \le \frac{C}{(|x_2|^2+|n|^2)^{\frac{m+1}{2}}}.$$

Let us turn to the series $\sum_n' K(n,x_2)$. It can be shown that the Fourier transforms of its partial sums

$$F\left\{\sum_{|n|<N}' K(n,x_2)\right\}$$

converge in $L_2(\Omega_2)$, and from this the required result follows.

With the aid of Lemma 2 the following assertion is established, which is used in the proof of the rule of multiplication of symbols.

Lemma 3. Let the function $a(x_1,x_2)$ have the following properties: 1) $a(x_1,x_2)$ is periodic in the variables $x^{(1)},\dots,x^{(k)}$, with period

is equal to one; 2) $a(x_1,x_2)$ is continuous in the whole strip Ω; 3) as $x_2\to\infty$ the function $a(x_1,x_2)$ tends to a constant uniformly with respect to x_1.

Then the operator $(aA_{km}-A_{km}a)$ is completely continuous in the space $L_2(\Omega)$.

The theorem given below is a consequence of Lemma 3 and the interpolation theorem of M. A. Krasnosel' skii ([4]).

Theorem 3 (multiplication rule for symbols). Let the symbols of the singular operators A and B be respectively $\Phi_A(x_1,x_2;\theta)$ and $\Phi_B(x_1,x_2;\theta)$, and let the symbol of the operator C be equal to the product $\Phi_A(x,\theta)\cdot\Phi_B(x,\theta)$. Suppose that the coefficients $a_{km}(x_1,x_2)$ and $b_{km}(x_1,x_2)$ in the expansions

$$\Phi_A(x_1,x_2;\theta)=\sum_{k=0}^{\infty} a_{km}(x_1,x_2)Y_{km}(\theta),$$

$$\Phi_B(x_1,x_2;\theta)=\sum_{k=0}^{\infty} b_{km}(x_1,x_2)Y_{km}(\theta)$$

possess properties 1)–3) of the function $a(x_1, x_2)$ from Lemma 3. Then the operator $AB - C$ is completely continuous in $L_p(\Omega)$, $1 < p < \infty$.

Now, by the usual method one obtains Noether theorems both for a single singular equation and for systems of equations (2). The question of the index of singular operators on an infinite cylinder is reduced to the solved question of the index of such operators on the torus ([5]).

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REFERENCES

[1] A. P. Calderon, A. Zygmund, *Studia math.*, **14**, 1, 249 (1955).
[2] S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Moscow, 1962.
[3] A. P. Calderon, A. Zygmund, *Acta math.*, **88**, 1–2, 85 (1952).
[4] M. A. Krasnosel' skii, DAN, **131**, No. 2, 246 (1960).
[5] M. F. Atiyah, I. M. Singer, *Bull. Am. Math. Soc.*, **69**, 422 (1963).

Note: Figure translations are in progress. See original paper for figures.