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Abstract

Full Text

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ASYMPTOTICS OF SOLUTIONS OF ALMOST LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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1°. In the present note we state theorems that generalize and refine the author' s results (1) concerning questions of asymptotic equivalence of solutions of the systems

$$x' = Ax + F(t, x); \tag{1}$$
$$y' = Ay. \tag{2}$$

Here A is a square matrix of order n with constant coefficients; $x, y, F(t, x)$ are n-dimensional vectors; $F(t, x)$ is defined for $t \geq t_0$ and arbitrary x;

$$F(t, 0) = 0; \tag{3}$$

$$|F(t, x_1) - F(t, x_2)| \leq g(t)|x_1 - x_2|, \tag{4}$$

where $g(t)$ is a nonnegative function.

2°. Obviously, the fraction $|x(t) - y(t)|/|y(t)|$ may be taken as a measure of the closeness of the vectors $x(t)$ and $y(t)$. We shall call this fraction the deviation of x from y, or simply the deviation. If the deviation of x from y tends to 0 as $t \to \infty$, then we shall call the vectors $x(t)$ and $y(t)$ analogous.

It is easy to see that the ratio of the lengths of analogous vectors tends to unity as $t \to \infty$, and the differences between the direction cosines (in the case of real x and y) tend to zero. Clearly, this occurs the more rapidly, the more rapidly the deviation tends to zero.

3°. Let $\omega_1 < \omega_2 < \cdots < \omega_s$ be all the distinct real parts of the eigenvalues of the matrix A. Consider those Jordan blocks in the Jordan form of A whose diagonal contains an eigenvalue with real part ω_k. Denote by m_{k+1} the order of the largest of these blocks.

Take any nonnegative number α, and let ω_0 denote an arbitrary number smaller than $\omega_1 - \alpha$: $\omega_0 < \omega_1 - \alpha$. For each $k = 1, 2 \ldots, s$, define the index $\tilde{k}$ by means of the inequalities

$$\omega_k - \alpha > \omega_{\tilde{k}-1}, \qquad \omega_k - \alpha \le \omega_{\tilde{k}}.$$

Obviously, the index $\tilde{k}$ is determined uniquely from this. In particular, if $\alpha = 0$, then $\tilde{k} = k$.

Denote by $m^0_{\tilde{k}}$ the number defined by the conditions

$$m^0_{\tilde{k}} = \begin{cases} 0, & \text{if } \omega_k - \alpha < \omega_{\tilde{k}}, \\ m_{\tilde{k}}, & \text{if } \omega_k - \alpha = \omega_{\tilde{k}}. \end{cases}$$

Since for $\alpha = 0$ one has $\tilde{k} = k$ and $\omega_k - \alpha = \omega_{\tilde{k}}$, in this case $m^0_{\tilde{k}} = m_k$.

4°. **Theorem 1.** Let α and β be arbitrary real numbers, with $\alpha > 0$. Suppose that

$$\int_{t_0}^{\infty} e^{\alpha\tau} \tau^{\beta} g(\tau)\, d\tau < +\infty.$$

Then there exists a topological mapping Φ of the space (x) onto the space (y) with the following properties: a) Φ and Φ^{-1} satisfy the Lipschitz condition; b) through the points corresponding under Φ at the instant $t = t^*$, where t^* is sufficiently large, there pass solutions of systems (1) and (2) that are analogous and have deviation

$$o\left(e^{-\alpha t} t^{m^0_{\tilde{k}} - \beta}\right).$$

Theorem 2. Let, for some nonnegative number β,

$$\int_{t_0}^{\infty} \tau^{\beta} g(\tau)\, d\tau < +\infty. \tag{5}$$

Then there exists a homeomorphism Φ mapping the space (x) onto the space (y) and having the following properties: a) Φ and Φ^{-1} satisfy a Lipschitz condition; b) the solutions of systems (1) and (2) passing at $t = t^*$, where t^* is sufficiently large, through points corresponding under Φ, have identical exponents; c) for every index k for which $\beta \ge m_k$, the solutions of systems (1) and (2) with exponents ω_k, passing at the initial instant through points corresponding under Φ, are analogous, and their deviation is $o(t^{m_k - \beta})$ as $t \to \infty$.

5°. A simple consequence of Theorem 1 is the criterion of V. A. Yakubovich [2], which guarantees the existence of such a homeomorphic mapping of the space (x) onto the space (y) that, through corresponding points at the initial instant, there pass solutions whose difference tends to zero as $t \to \infty$. From Theorem 2 it is not difficult to obtain one of the reducibility theorems [3] of the same author. By the standard method, using the principle of linear inclusion [5], one can derive from Theorem 2 a proposition very close to the theorem of A. Wintner and P. Hartman [4].

If the vector $F(t,x)$, instead of satisfying requirement (4), satisfies the inequality $|F(t,x)| \leq g(t)|x|$ and condition (5) is fulfilled, then every solution of system (1) with exponent ω_k, where k is such that $m_k \leq \beta$, is analogous to some solution of system (2), and their deviation is $o(t^{m_k-\beta})$ as $t \to \infty$.

6°. To illustrate the sharpness of the results obtained, let us consider examples.

Example 1. Let the systems be given by

$$\dot{x}_1 = -x_1 + g(t)x_4; \qquad \dot{x}_2 = x_1 - x_2; \qquad \dot{x}_3 = -x_3; \qquad \dot{x}_4 = x_3 - x_4; \quad (6)$$

$$\dot{y}_1 = -y_1; \qquad \dot{y}_2 = y_1 - y_2; \qquad \dot{y}_3 = -y_3; \qquad \dot{y}_4 = y_3 - y_4, \quad (7)$$

where $g(t)$ is some positive function.

If

$$\int_{t_0}^{\infty} \tau g(\tau)\, d\tau < +\infty,$$

then, by Theorem 2, every solution of (6) is analogous to some solution of system (7). Suppose that

$$\int_{t_0}^{\infty} \tau g(\tau)\, d\tau = \infty.$$

Consider the solution $x(t)$ of system (6) with initial conditions at $t = 0$ $(0;0;1;0)$. Obviously, the coordinates of this solution are given by the formulas

$$x_1 = e^{-t}\int_0^t \xi g(\xi)\, d\xi; \qquad x_2 = e^{-t}\int_0^t d\tau \int_0^\tau \xi g(\xi)\, d\xi; \qquad x_3 = e^{-t}; \qquad x_4 = te^{-t}.$$

Put

$$\int_0^t \xi g(\xi)\,d\xi = \varphi(t).$$

By assumption, $\varphi(t) \to \infty$ as $t \to \infty$, and since $g(\xi) > 0$, $\varphi(t)$ increases monotonically.

For any solution $y(t) = \{y_1(t); y_2(t); y_3(t); y_4(t)\}$ of system (7), the inequality

$$|x(t) - y(t)| \geq |x_2(t) - y_2(t)| = e^{-t}\left|\int_0^t \varphi(\tau)\,d\tau - at - b\right| \geq$$

$$\geq e^{-t}\left\{\int_0^t \varphi(\tau)\,d\tau - t\left|a + \frac{b}{t}\right|\right\}.$$

(Here a and b are arbitrary constants.)

Using the monotonicity and positivity of $\varphi(t)$, we can write

$$\int_0^t \varphi(\tau)\,d\tau > \int_{t/2}^t \varphi(\tau)\,d\tau > \frac{t}{2}\,\varphi\left(\frac{t}{2}\right).$$

Hence

$$\frac{|x(t) - y(t)|}{|y(t)|} \geq \frac{1}{|y(t)|}\,e^{-tt}\left\{\frac{1}{2}\,\varphi\left(\frac{t}{2}\right) - \left|a + \frac{b}{t}\right|\right\}.$$

Since $|y(t)|$ "grows" no faster than e^{-tt}, and $\varphi(t) \to \infty$ as $t \to \infty$, the deviation of the solution $x(t)$ of system (6) under consideration from any solution $y(t)$ of system (7) increases without bound as $t \to \infty$. Consequently, for $x(t)$ there is no analogous solution of system (7).

Thus, without imposing some additional restrictions on the matrix A or the vector $F(t, x)$, the conditions of Theorem 2 cannot be weakened.

Example 2. Consider the systems

$$\dot{x}_1 = -2x_1 + g(t)x_4; \qquad \dot{x}_2 = x_1 - 2x_2; \qquad \dot{x}_3 = -x_3; \qquad \dot{x}_4 = x_3 - x_4; \quad (8)$$

$$\dot{y}_1 = -2y_1; \qquad \dot{y}_2 = y_1 - 2y_2; \qquad \dot{y}_3 = -y_3; \qquad \dot{y}_4 = y_3 - y_4. \qquad (9)$$

If

$$\int_{t_0}^{\infty} e^{\tau} g(\tau)\, d\tau < +\infty,$$

then, by Theorem 1, any solution of system (8) is analogous to some solution of system (9), and their deviation is $o(e^{-t})$. Let

$$\int_{t_0}^{\infty} e^{\tau} g(\tau)\, d\tau = \infty.$$

Then it can be shown, just as was done in Example 1, that the deviation of the solution $x(t)$ of system (8), passing at $t = 0$ through the point (0; 0; 1; 0), from any solution of system (9) is a quantity infinitely large in comparison with e^{-t}.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.