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Abstract

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MATHEMATICAL PHYSICS

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ON A TWO-LAYER THERMOCONVECTIVE PROBLEM

(Presented by Academician A. A. Dorodnitsyn on 14 III 1963)

We pose the problem:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}, \quad y > 0,\ t > 0; \tag{1a}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} + \nu\frac{\partial u}{\partial x}, \quad -1 < y < 0,\ x > 0,\ t > 0; \qquad (1\ )$$

$$u\big|_{t=0} = 0, \quad u\big|_{x=0} = f(t), \quad -1 < y < 0; \qquad (1\ )$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=-1} = 0, \quad \lim_{x^2+y^2\to\infty} u = 0; \qquad (1\ )$$

$$u\big|_{y=0+} = u\big|_{y=0-}, \quad \lambda\left.\frac{\partial u}{\partial y}\right|_{y=0+} = \left.\frac{\partial u}{\partial y}\right|_{y=0-}. \qquad (1\ )$$

Here a^2 and λ are the ratios of the coefficients of thermal diffusivity and thermal conductivity of the two media, respectively, and ν is the convective parameter.

The function $f(t)$ is subject to conditions sufficient for the existence of the Laplace transform (it is absolutely integrable on any interval $0 \le t \le A$, and $|f(t)| < e^{ct}$ for large values of t, where c is a constant).

We seek a solution of problem (1) that is continuous in the domain $y > 0$ and allows a discontinuity on the line $t = x/\nu$ in the domain $-1 < y < 0$*.

It is not hard to see that the solution of problem (1) is identically equal to zero for $x/\nu > t$, and $u|_{x/\nu\to t-0} = 0$ in the domain $y > 0$.

Make the change of variables:

$$\xi = t - \frac{x}{\nu}, \quad \eta = \frac{x}{\nu}.$$

Problem (1) becomes the following:

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{1}{a^2}\frac{\partial u_1}{\partial \xi}, \quad y > 0, \quad \xi > 0; \tag{2a}$$

$$\frac{\partial^2 u_2}{\partial y^2} = \frac{\partial u_2}{\partial \eta}, \quad -1 < y < 0, \quad \eta > 0; \tag{2 }$$

$$u_1\big|_{\xi=0} = 0, \quad u_2\big|_{\eta=0} = f(\xi); \tag{2 }$$

$$\frac{\partial u_2}{\partial y}\bigg|_{y=-1} = 0, \quad \lim_{y^2+\eta^2\to\infty} u_1 = 0, \quad (i = 1, 2); \tag{2 }$$

$$u_1 = u_2, \quad \lambda\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} \quad \text{for } y = 0. \tag{2 }$$

Let $G_1(y, y', \xi - \xi')$ and $G_2(y, y', \eta - \eta')$ be the Green' s functions of the second boundary-value problem for equations (2a) and (2), respectively,

$$G_1 = E(y - y', a^2(\xi - \xi')) + E(y + y', a^2(\xi - \xi')),$$

$$G_2 = \sum_{n=-\infty}^{\infty} \left[E(y - y' + 2n, \eta - \eta') + E(y + y' + 2n, \eta - \eta')\right],$$

$$E(x, y) = \frac{1}{2\sqrt{\pi y}} \exp\left(-\frac{x^2}{4y}\right).$$

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* Such a problem arises, for example, in determining the temperature field of a reservoir under thermal injection, if one takes into account heat conduction of the rocks and of the reservoir in the vertical direction, while neglecting heat conduction in the horizontal direction. This makes it possible, in particular, to establish the admissibility of the “lumped-capacity”scheme adopted in describing the temperature field of a reservoir in a number of works (1-3).

Denoting

$$\left.\frac{\partial u_2}{\partial y}\right|_{y=0} = \varphi(\xi, \eta),$$

we shall have:

$$u_1(y, \xi, \eta) = \frac{1}{\lambda} \int_0^{\xi} G_1(y, 0, \xi - \xi')\varphi(\xi', \eta)\, d\xi'; \tag{3}$$

$$u_2(y, \xi, \eta) = f(\xi) \int_{-1}^{0} G_2(y, y', \eta)\, dy' - \int_0^{\eta} G_2(y, 0, \eta - \eta')\varphi(\xi, \eta')\, d\eta', \tag{4}$$

where

$$\int_{-1}^{0} G_2(y, y', \eta)\, dy' = 1.$$

Using the remaining, as yet unsatisfied, first condition (2d), we obtain an integral equation for φ:

$$\frac{k}{\sqrt{\pi}} \int_0^{\xi} \frac{\varphi(\xi', \eta)\, d\xi'}{\sqrt{\xi - \xi'}} = f(\xi) - \frac{1}{\sqrt{\pi}} \int_0^{\eta} \frac{\varphi(\xi, \eta')\, d\eta'}{\sqrt{\eta - \eta'}} - 4 \int_0^{\eta} \sum_{n=1}^{\infty} E(2n, \eta - \eta')\varphi(\xi, \eta')\, d\eta', \tag{5}$$

where $k = 1/\lambda a$.

In the usual way, equation (5) is reduced to an integro-differential equation. We shall reduce equation (5) to a Volterra integral equation of the second kind.

For this purpose, we apply to both sides the transform

$$\Phi(p, q) = pq \int_0^{\infty} \int_0^{\infty} \varphi(\xi, \eta) e^{-\xi p - \eta q}\, d\xi\, d\eta;$$

formally we obtain

$$\left(\frac{k}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right) \Phi(p, q) = F(p) - 4K(q)\Phi(p, q), \tag{6}$$

where

$$K(q) = \int_0^{\infty} \sum_{n=1}^{\infty} E(2n, \eta) e^{-\eta q}\, d\eta,$$

$$F(p) = p\int_0^\infty f(\xi)e^{-\xi p}\,d\xi;$$

computing the integral, we obtain

$$K(q) = \frac{1}{2\sqrt{q}} - \frac{e^{-2\sqrt{q}}}{1-e^{-2\sqrt{q}}}.$$

Thus, we have:

$$\Phi(p,q) = \frac{\sqrt{pq}}{\sqrt{p}+k\sqrt{q}}F(p) - \frac{2\sqrt{p}}{\sqrt{p}+k\sqrt{q}}\,\frac{e^{-2\sqrt{q}}}{1-e^{-2\sqrt{q}}}\Phi(p,q)^*. \tag{7}$$

Adding to both sides the quantity

$$\frac{2e^{-2\sqrt{q}}}{1-e^{-2\sqrt{q}}}\,\Phi,$$

we transform (7) to the form

$$\Phi(p,q) = \frac{\sqrt{pq}}{\sqrt{p}+k\sqrt{q}}\,\mathrm{th}\,\sqrt{q}\,F(p) + \frac{2\sqrt{q}}{\sqrt{p}+k\sqrt{q}}\,\frac{e^{-2\sqrt{q}}}{1+e^{-2\sqrt{q}}}\Phi(p,q). \tag{8}$$

* From this, Φ can be expressed explicitly and one can pass to the original φ; however, computation of φ is difficult because of the essential singularities that appear after inversion of Φ with respect to one of the variables.

Passing in (8) to the originals and carrying out all the calculations, we obtain the Volterra integral equation of the second kind:

$$\varphi(\xi,\eta) =$$

$$= \frac{d}{d\xi}\int_0^\xi f(\xi')\,m(\xi-\xi',\eta)\,d\xi' + \int_0^\xi\int_0^\eta h(\xi-\xi',\eta-\eta')\varphi(\xi',\eta')\,d\xi'\,d\eta', \tag{9}$$

where

$$m(\xi,\eta) = \frac{1}{\sqrt{\pi}\sqrt{\eta+k^2\xi}} + \frac{4}{\sqrt{\pi}}\sum_{n=1}^\infty(-1)^n\int_0^\eta E(2n,\eta-u)\,\frac{n\,du}{(\eta-u)\sqrt{u+k^2\xi}},$$

$$h(\xi,\eta)=\frac{k}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n^2\eta+k^2\xi^2-n^2}{\sqrt{\xi\eta}(\eta+k^2\xi)^3}e^{-n^2/\eta}+\right.$$

$$\left.+nk\sqrt{\pi}\,\frac{3(\eta+k^2\xi)-2n^2}{(\eta+k^2\xi)^{7/2}}\exp\left(-\frac{n^2}{\eta+k^2\xi}\right)\operatorname{erfc}\frac{nk\sqrt{\xi}}{\sqrt{\eta(\eta+k^2\xi)}}\right).$$

The kernel $h(\xi,\eta)$ has the form $\frac{1}{\sqrt{\xi}}N(\xi,\eta)$, where N is a continuous function; therefore the general theory of the Volterra equation of the second kind is applicable to equation (9). From the properties of the solution of equation (9) it follows that φ satisfies all conditions necessary for the existence of its double Laplace transform. Therefore all operations performed on equation (5) are legitimate and, consequently, equation (5) is equivalent to equation (9). Thus we obtain the unique solution of problem (2). Returning to the old variables, we obtain the solution of problem (1).

For small values of ξ, already the zeroth approximation, which may be taken as the free term of equation (9), gives a good approximation to the solution.

In applications, of particular interest is the value of u_2 on the "temperature front" ($t=x/v$), i.e. for $\xi=0$. From equation (9), for $\xi=0$ ($f(\xi)\equiv 1$), we have

$$\varphi(\eta)=\frac{1}{\sqrt{\pi\eta}}+\frac{4}{\sqrt{\pi}}\sum_{n=1}^{\infty}(-1)^n\int_0^{\eta}E(2n,\eta-u)\,\frac{n\,du}{(\eta-u)\sqrt{u}}=$$

$$=\frac{1}{\sqrt{\pi\eta}}+\frac{2}{\sqrt{\pi\eta}}\sum_{n=1}^{\infty}(-1)^n\exp\left(-\frac{n^2}{\eta}\right).$$

Substituting the value of $\varphi(\eta)$ into (4), we find u_2 on the temperature front:

$$u_2(y,0,\eta)=$$

$$=1-4\int_0^{\eta}\sum_{n=-\infty}^{\infty}E(y+2n,\eta-\eta')\left[E(0,\eta')+2\sum_{n=1}^{\infty}(-1)^{nE}(2n,\eta')\right]d\eta'.$$

Computing the integral (for example, by means of the Laplace transform), we obtain

$$u_2(y,0,\eta)=1-\sum_{n=0}^{\infty}(-1)^n\left(\operatorname{erfc}\frac{1-y+2n}{2\sqrt{\eta}}+\operatorname{erfc}\frac{1+y+2n}{2\sqrt{\eta}}\right).$$

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Note: Figure translations are in progress. See original paper for figures.