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Abstract

Full Text

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ON ω-MAPPINGS AND INVERSE SPECTRA

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All spaces considered below are assumed to be Hausdorff, and the mappings continuous.

Theorem 1. *If there is a countable system of mappings $f_i : X \to R_i$, $i = 1, 2, \ldots$, of a normal space X with $\dim X = r$ onto metric spaces (respectively, onto metric spaces with a countable base) R_i, then there exists a metric space (respectively, a metric space with a countable base) S with $\dim S \leq r$ and mappings $g : X \to S$ and $h_i : S \to R_i$ such that $h_i \cdot g = f_i$, $i = 1, 2, \ldots$.*

Remark 1. Theorem 1 generalizes Lemma 4 of ([1]), for if X is bicompact, then S is compact.

Theorem 2. *If a normal space X with $\dim X = r$ is the limit of some spectrum of metric spaces (respectively, of metric spaces with a countable base), then X is also the limit of a spectrum of r-dimensional, in the sense of $\dim$, metric spaces (metric spaces with a countable base).*

Definition 1. A space X will be called **spectrally decomposable with respect to a class of spaces $\mathfrak{M}$** (a **spectral $\mathfrak{M}$-space**) if X is the limit of some inverse spectrum $S = \{X_\alpha, \mathfrak{F}_\alpha^\beta\}$, $\alpha \in \mathfrak{A}$, where all X_α belong to the class $\mathfrak{M}$.

Definition 2. A class of spaces $\mathfrak{M}$ will be called: 1) **monotone**, if from $Y \in \mathfrak{M}$ and from $A \subset Y$ it follows that $A \in \mathfrak{M}$; 2) **closed with respect to finite multiplication**, if from Y and $Z \in \mathfrak{M}$ it follows that $Y \times Z \in \mathfrak{M}$; 3) **sufficiently broad**, if $\mathfrak{M}$ contains all metric spaces with a countable base.

A class of spaces $\mathfrak{M}$ satisfying conditions 1), 2) or, respectively, conditions 1), 2), 3), will be called a **class of type (1–2)** or, respectively, a **class of type (1–3)**.

Examples of classes of type (1–3) are: the classes of spaces with the first axiom of countability, spaces with a countable base, metric spaces, strongly metrizable metric spaces ([2]), metric spaces with a countable base.

Theorem 3. *If a Hausdorff space X is the limit of a spectrum $S = \{X_\alpha, \mathfrak{F}_\alpha^\beta\}$, $\alpha \in \mathfrak{A}$, of Hausdorff spaces X_α with projections $\mathfrak{F}_\alpha^\beta$ that are, in general, mappings "onto," then for every Hausdorff one-point extension $\widetilde{X} = X \cup x_0$ of the space X there exists an index $\alpha_0 \in \mathfrak{A}$ such that the projection $\mathfrak{F}_{\alpha_0} : X \to X_{\alpha_0}$ cannot be extended to a continuous mapping of the extension $\widetilde{X}$ into the space X_{α_0}.*

Definition 3. A mapping $f : X \to Y$ with respect to some covering ω of the space X is called: 1) an **ω-mapping** (respectively, a **finite ω-mapping**, a **countable ω-mapping**) if for every point $y \in Y$ there exists a neighborhood O_y such that the set $f^{-1}(O_y)$ is wholly contained in one element (in a finite number of elements, in a countable number of elements) of the cover-

ω; 2) a **weak ω-mapping** (respectively, a **finite weak ω-mapping**, a **countable weak ω-mapping**) if for each point $y \in Y$ the set $f^{-1}(y)$ is wholly contained in one element (respectively in a finite, in a countable number of elements) of the cover ω.

Remark 2. Every bicompact (finally compact) mapping $f : X \to Y$ is a finite (countable) weak ω-mapping for any cover ω of the space X; if, moreover, the mapping f is closed, then it is a finite (countable) ω-mapping for any cover ω of the space X.

Theorem 4. In order that a T_1-space X have a homeomorphic mapping onto an everywhere dense subset of the limit of some spectrum $S = \{X_\alpha, \mathfrak{F}_\alpha^\beta\},\ \alpha \in \mathfrak{A}$, of spaces X_α belonging to a class $\mathfrak{M}$ of type (1—2), it is necessary and sufficient that for every point-binary cover* ω of the space X there exist an ω-mapping $f_\omega : X \to X_\omega$, where $X_\omega \in \mathfrak{M}$.

Theorem 5. In order that a T_1-space X have a one-to-one and continuous mapping onto the limit of some spectrum $S = \{X_\alpha, \mathfrak{F}_\alpha^\beta\},\ \alpha \in \mathfrak{A}$, of spaces X_α from some class of Hausdorff spaces $\mathfrak{M}$ of type (1—2), it is sufficient that for every cover ω of the space X there exist a weak ω-mapping $g_\omega : X \to Y_\omega$, where $Y_\omega \in \mathfrak{M}$.

Theorem 6. In order that a T_1-space X be a spectral $\mathfrak{M}$-space, where $\mathfrak{M}$ is a class of Hausdorff spaces of type (1—2), it is sufficient that two conditions be fulfilled: 1) for every point-binary cover ω of its own, the space X have an ω-mapping $f_\omega : X \to X_\omega$, where $X_\omega \in \mathfrak{M}$; 2) for every cover ω of its own, the space X have a weak ω-mapping $g_\omega : X \to Y_\omega$, where $Y_\omega \in \mathfrak{M}$.

In particular, a T_1-space X will be a spectral $\mathfrak{M}$-space, where $\mathfrak{M}$ is a class of Hausdorff spaces of type (1—2), if for every cover ω of its own the space X has an ω-mapping into some space $X_\omega \in \mathfrak{M}$.

Since every: a) paracompact, b) strongly paracompact, c) regular finally compact, d) bicompact Hausdorff space X, for any cover ω of its own, has an ω-mapping respectively onto: a) a metric space, b) a strongly metrizable metric space, c) a metric space with a countable base, d) a compact metric space (see ([3,4])), it follows that

Theorem 7. a) Every paracompactum is spectrally decomposable with respect to the class of metric spaces; b) every strongly paracompact Hausdorff space is spectrally decomposable with respect to the class of strongly metrizable metric spaces; c) every regular finally compact space is spectrally decomposable with respect to the class of metric spaces with a countable base**.

From Theorems 2 and 7 it follows that

Theorem 8. a) Every paracompactum that is r-dimensional in the sense of dim is the limit of some spectrum of r-dimensional, in the sense of dim, metric spaces; b) every r-dimensional, in the sense of dim, regular finally compact space is the limit of some spectrum of r-dimensional metric spaces with a countable base.

As a consequence of Theorem 8 we obtain Theorem 1 from ([1]).

Theorem 9. In order that a strongly paracompact space X have $\dim X \leq r$, it is necessary and sufficient that the space X be spectrally decomposable with respect to the class of r-dimensional, in the sense of dim, metric spaces***.

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* A point-binary cover is any cover of the form $\{Ox,\ X \setminus x\}$, where Ox is a neighborhood of the point x.

** Parts a) and c) of this theorem were also proved by V. Ponomarev.

*** The sufficiency of the formulated condition was also proved by Yu. M. Smirnov.

Theorem 10. In order that a Hausdorff (respectively, completely regular) space X be spectrally decomposable with respect to the class of Hausdorff spaces $\mathfrak{M}$ of type (1–2), it is necessary and sufficient that: a) for each of its point-binary covers ω, the space X have an ω-map $f_\omega : X \to X_\omega$, where $X_\omega \in \mathfrak{M}$; b) for any of its Hausdorff (respectively, completely regular) one-point extensions $X \cup x_0$, the space X have such a map $f_{x_0} : X \to X_{x_0}$, $X_{x_0} \in \mathfrak{M}$, which cannot be extended to a continuous map of the extension $X \cup x_0$ into the space X_{x_0}.

Theorem 11. In order that a completely regular space X be spectrally decomposable with respect to the class of metric spaces with a countable base, it is necessary and sufficient that one of the following conditions be fulfilled: a) for any point $x_0 \in \beta X \setminus X^*$ there exists a map f_{x_0} of the space X onto a metric space with a countable base X_{x_0}, which cannot be extended to a continuous map of the set $X \cup x_0 \subseteq \beta X$ into the space X_{x_0}; b) every point $x_0 \in \beta X \setminus X$ is a set of type G_δ (has a countable pseudocharacter) in the set $X \cup x_0 \subseteq \beta X$; c) for any point $x_0 \in \beta X \setminus X$ there exists a countable normal** star-finite cover of the space X, the closures (in βX) of whose elements do not meet the point x_0.

Theorem 12. A normal space X is spectrally decomposable with respect to the class of metric spaces with a countable base if and only if it has such a map f onto some metric space with a countable base Y that every set $f^{-1}(y)$, $y \in Y$, is spectrally decomposable with respect to the class of metric spaces with a countable base.

Theorem 13. In order that a completely regular space X be spectrally decomposable with respect to the class of metric spaces (respectively, strongly metrizable metric spaces, metric spaces with a countable base), it is sufficient that for each of its covers ω the space X have a countably weak ω-map onto some space of the corresponding class.

In particular, if the space X has a bicompact or even finally compact map onto some metric space (respectively, onto a strongly metrizable metric space, a metric space with a countable base), then it is spectrally decomposable with respect to the corresponding class of metric spaces.***

Theorem 14. a) A completely regular space X is spectrally decomposable with respect to the class of metric spaces if and only if for any point $x_0 \in \beta X \setminus X$ there exists such a map f of the space X onto a metric space X_{x_0} which cannot be extended to a continuous map of the set $X \cup x_0 \subseteq \beta X$ onto the space X_{x_0};
b) a completely regular space X is spectrally decomposable with respect to the class of metric spaces if and only if for any point $x_0 \in \beta X \setminus X$ there exists such a normal locally finite in X cover whose closures of elements in βX do not meet the point x_0.

Theorem 15. The classes of spectrally paracompact (spectrally finally compact) and spectrally metric spaces (spectrally metric spaces with a countable base) coincide****, and coincide

* By βX everywhere is meant the maximal (Stone–Čech) bicompact extension of the space X.

** See (5), footnote on p. 73.

*** We note that a finally compact space has a finally compact map simply to a point.

**** That is (see Definition 1), if a space X is the limit of a spectrum of paracompact (finally compact) spaces, then it is the limit of a spectrum of metric spaces (with a countable base).

with the class of spaces that are closed subsets of direct products of paracompact (finally compact), or, what is the same, metrizable spaces (with a countable base).

If a spectrally paracompact space X is normal and $\dim X = r$, then X may be regarded as a closed subset of a product of r-dimensional metrizable spaces.

Theorem 16. a) A completely regular space X is functionally closed if and only if it is spectrally finally compact.
b) A completely regular space X is complete in the sense of Dieudonné if and only if it is spectrally paracompact.

We shall call a spectrally paracompact (spectrally finally compact) space $\widetilde{X}$ a **spectrally paracompact (spectrally finally compact) extension of the space** X, if X is everywhere dense in $\widetilde{X}$ and in $\widetilde{X}$ there is no spectrally paracompact (spectrally finally compact) subset X' such that

$$X \subseteq X' \subset \widetilde{X}.$$

Theorem 17. a) The completion of a completely regular space X with respect to its maximal uniform structure coincides with that spectrally paracompact extension μX of the space X to which every mapping of the space X into a metrizable (spectrally metrizable) space extends.
b) The Hewitt (i.e., maximal functionally closed) extension υX ([5]), p. 72) of a completely regular space X coincides with that spectrally finally compact extension of the space X to which every mapping of the space X into a metrizable space with a countable base (into a spectrally finally compact space) extends.

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Note: Figure translations are in progress. See original paper for figures.