ON THE PROBLEM OF SOLVING THE COAGULATION EQUATION FOR RAIN DROPS WITH CONDENSATION TAKEN INTO ACCOUNT

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Abstract

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ON THE PROBLEM OF SOLVING THE COAGULATION EQUATION FOR RAIN DROPS WITH CONDENSATION TAKEN INTO ACCOUNT

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The paper investigates the problem of the spectrum of drops in an ascending air current under the assumption that the collision probability is a linear function of the volumes of the colliding drops. The influence of the condensational growth of drops on the spectrum is considered.

Without taking condensation into account, the form of the spectrum is determined by the dependence of the collision probability on the volumes of the colliding drops. Since at present the question of the collision probability of cloud drops cannot be considered solved, it is expedient to seek solutions of the kinetic equation under definite assumptions concerning the collision probability. N. N. Tunitskii ([1]) proposed a method for solving the kinetic equation with a constant collision probability, which is acceptable in the study of Brownian coagulation. Since in clouds the principal role is played by gravitational coagulation, caused by the dependence of $\dot{z}$—the falling velocity of drops—on their radius r (for $r \ll 4 \cdot 10^{-3}$ cm, $\dot{z} \sim r^2$, while for $r \simeq 0.1$ cm, $\dot{z} \sim \sqrt{r}$), then over a fairly wide interval one may take $\dot{z} \sim r$, and β—the collision probability of particles of radii r_1 and r_2—to be proportional to $(r_1 + r_2)^2 |r_1 - r_2|$.

However, for collisions of drops of nearly equal sizes this result is inapplicable. If $r_1 \simeq r_2 \lesssim 10^{-3}$ cm, then, as was shown by V. G. Levich ([2]), collisions are governed by a turbulent diffusion mechanism, according to which the collision probability is proportional to $(r_1 + r_2)^3$. In collisions of larger drops there is interaction of the hydrodynamic flows surrounding the drops, and, as follows from the calculations of Pearcey and Hill ([3]), if the collision probability is described by the formula $\beta = k(r_1 + r_2)^2 |\dot{z}(r_1) - \dot{z}(r_2)|$, then the coefficient k may be considered constant except in the region $r_1 \simeq r_2$. Thus, for example, for the case $r_1 = 7 \cdot 10^{-3}$ cm, $r_2 = 0.99 r_1$, the value of k should be increased, in comparison with the case $r_1 \sim r_2$, by approximately a factor of 100. This makes it possible to choose β approximately in the form

$$\beta(u, v) = b(u + v), \tag{1}$$

where u, v are the volumes of the colliding drops, and the constant $b \simeq 6 \cdot 10^3$ sec^{-1}.

With this choice of β, the kinetic equation is substantially simplified:

$$\frac{\partial n(v,t)}{\partial t} = \frac{1}{2} bv \int_0^v n(u,t)\, n(v-u,t)\, du -$$

$$- bvn(v,t) \int_0^\infty n(u,t)\, du - bn(v,t) \int_0^\infty n(u,t)\, u\, du, \tag{2}$$

where $n(v,t)$ is the number of drops of volume v at time t. Next we introduce the following notation:

$$N(t) = N_0(1-\tau), \tag{3}$$

where $N(t)$ is the total number of drops at time t in 1 cm^3, if N_0 is their number at the initial instant,

$$n(v,t) = N_0 \varphi(v,\tau). \tag{4}$$

Let us note that formula (3) takes the form

$$\int_0^\infty \varphi(v,\tau)\, dv = 1 - \tau. \tag{5}$$

Introduce v_0—the volume of the liquid phase in 1 cm^3, divided by N_0. Obviously, v_0 does not change due to coagulation,

$$\int_0^\infty v\varphi(v,\tau)\, dv = v_0. \tag{6}$$

The equation for the change in the total number of drops per unit volume

$$\frac{d}{dt} N(t) = -\frac{1}{2} b \int_0^\infty \int_0^\infty (u+v) n(u,t) n(v,t)\, du\, dv \tag{7}$$

with account of expressions (3), (5), and (6) makes it possible to find $\tau(t)$:

$$1 - \tau(t) = \exp\{-N_0 b v_0 t\}. \tag{8}$$

Thus, our problem reduces to solving the kinetic equation

$$\frac{\partial\varphi(v,\tau)}{\partial\tau} = \frac{1}{2}\,\frac{v}{v_0(1-\tau)}\int_0^v \varphi(u,\tau)\varphi(v-u,\tau)\,du - \frac{\varphi(v,\tau)}{1-\tau} - \frac{v}{v_0}\varphi(v,\tau) \quad (9)$$

with the initial distribution, which we choose in the form

$$\varphi(v,0) = \frac{1}{v_0}e^{-v/v_0} \quad (10)$$

and the additional condition (5).

We use the Laplace transform:

$$(1-\tau)\Phi(p,\tau) = \int_0^\infty e^{-pv}\varphi(v,\tau)\,dv. \quad (11)$$

Then the image of equation (9) will be the equation

$$\frac{\partial\Phi}{\partial\tau} + \frac{1}{v_0}(\Phi-1)\frac{\partial\Phi}{\partial p} = 0 \quad (12)$$

with the initial condition

$$\Phi(p,0) = \frac{1}{1+pv_0} \quad (13)$$

and the additional condition

$$\Phi(0,\tau) = 1. \quad (14)$$

From the characteristic equations

$$d\tau = \frac{v_0 dp}{\Phi-1} = \frac{d\Phi}{0} \quad (15)$$

we find two independent first integrals:

$$\Phi = C_1, \quad (16)$$

$$pv_0 - (\Phi-1)\tau = C_2. \quad (17)$$

Using the initial condition (13), we arrive at an equation for finding $\Phi(p,\tau)$,

$$pv_0 = (\Phi - 1)\tau + \frac{1-\Phi}{\Phi}. \tag{18}$$

Of the two roots of this equation we choose the one decreasing with increasing p, which corresponds to the condition of boundedness of the distribution function. We obtain

$$\Phi(p,\tau) = \frac{1}{2\tau}\left[pv_0 + 1 + \tau - \sqrt{(pv_0 + 1 + \tau)^2 - 4\tau}\right], \tag{19}$$

therefore,

$$\varphi(v,\tau) = \frac{1-\tau}{v\sqrt{\tau}}\, I_1\left(2\frac{v}{v_0}\sqrt{\tau}\right) \exp\left\{-(1+\tau)\frac{v}{v_0}\right\}. \tag{20}$$

As $\tau \to 1$, for $v \gg v_0$, in accordance with the asymptotic form of the Bessel function $I_1(x) \simeq e^x/\sqrt{2\pi x}$, we obtain

$$\lim_{\tau\to 1} \frac{\varphi(v,\tau)}{1-\tau} = \frac{1}{2}\sqrt{\frac{v_0}{\pi}}\, v^{-3/2}, \tag{21}$$

i.e., the evolution of the spectrum proceeds from exponential to power-law. Note that the asymptotic behavior of the spectrum described by formula (21) does not depend on the initial distribution, provided the value v_0 is preserved.

1. Thus, if the initial distribution is taken to be

$$\varphi(v,0) = \delta(v - v_0), \tag{22}$$

then instead of formula (18) we obtain

$$pv_0 = (\Phi - 1)\tau - \ln\Phi. \tag{23}$$

Since effectively $p \sim (1-\tau)/v_0$, then for $1 - \tau \ll 1$, taking condition (14) into account, we may set $\Phi \simeq 1$, and therefore, expanding the logarithm in a series and retaining only the first term, we again arrive at formula (18). This result makes it possible to solve approximately the problem of finding the spectrum of coagulating drops with allowance for their condensational growth. In this case, to the left-hand side of the kinetic equation one must add

$$\frac{\partial}{\partial v}\left[n(v,t)\left(\frac{dv}{dt}\right)_{\text{cond}}\right],$$

where $\left(\frac{dv}{dt}\right)_{\text{cond}}$ is the rate of condensational growth of a drop of volume v.

Since condensation is significant mainly for small times, when the distribution is close to a δ-functional one, $(dv/dt)_{\text{cond}}$ may be replaced by dv_0/dt.

Note that now, instead of formula (8), one should write

$$1-\tau=\exp\left\{-N_0 b\int_0^t v_0(t)\,dt\right\}. \tag{24}$$

The form of the interpolating kinetic equation with allowance for condensation will be

$$\frac{\partial F}{\partial\tau}+\frac{dv_0}{dt}\frac{pF}{1-\tau}+\frac{1}{v_0}(F-1)\frac{\partial F}{\partial p}=0, \tag{25}$$

where

$$(1-\tau)F(p,\tau)=\int_0^\infty e^{-pv}\varphi(v,\tau)\,dv \tag{26}$$

with the initial condition

$$F(p,0)=1, \tag{26'}$$

which corresponds to the choice $\varphi(v,0)=\delta(v)$, and with the additional condition

$$\frac{\partial F(0,\tau)}{\partial p}=-\frac{v_0(\tau)}{1-\tau}, \tag{27}$$

the presence of which is explained by the fact that we shall seek an approximate solution of the kinetic equation satisfying the prescribed function $v_0(\tau)$ in advance.

We shall seek the solution of equation (25) in the form

$$F(p,\tau)=e^{-pv_0(\tau)}G(p,\tau), \tag{28}$$

where the first factor determines the evolution of the spectrum due to condensational growth; then, for determining $G(p,\tau)$, we obtain an equation which, as follows from the solution obtained, for sufficiently large τ coincides with equation (12).

The general solution of equation (12) will be

$$p=(G-1)\frac{\tau}{v_0}+\Psi(G), \tag{29}$$

where $\Psi(G)$ is an arbitrary function of its argument. From conditions (26) and (27) it follows that G must satisfy the conditions

$$G(p, 0) = 1; \tag{30}$$

$$\frac{\partial G(0, \tau)}{\partial p} = -\frac{\tau}{1-\tau} v_0(\tau). \tag{31}$$

Taking condition (30) into account,

$$\Psi(1) = 0. \tag{32}$$

Consequently, as $\tau \to 1$, $\Psi(G) \sim G - 1$, if one admits the existence of a Taylor series for $\varphi(v, \tau)$ in powers of $v - v_0$, then, when $\varphi(v_0, \tau) \neq 0$,

$$G(p, \tau) \underset{p\to\infty}{\sim} \frac{1}{p}. \tag{33}$$

Therefore, in order to satisfy relation (29), one should take

$$\Psi(G) \sim \frac{1-G}{G}. \tag{34}$$

Let us note that this choice is equivalent to the assumption of an exponential initial distribution. However, as was indicated above, this choice is not essential in determining the spectrum as $\tau \to 1$. Since expression (29) is valid only for $1 - \tau \ll 1$, in order to satisfy relation (31) for any τ, instead of $\Psi(G)$ one should write

$$\frac{1-\tau+\tau^2}{v_0\tau} \frac{1-G}{G}.$$

Thus, we arrive at the equation for determining $G(p, \tau)$. Let us give the final result:

$$F(p, \tau) = \frac{v_0}{2\tau} e^{-pv_0} \left[p + \frac{1-\tau+2\tau^2}{v_0\tau} - \sqrt{\left(p + \frac{1-\tau+2\tau^2}{v_0\tau}\right)^2 - \frac{4}{v_0^2}(1-\tau+\tau^2)} \right]. \tag{35}$$

Thus, $\varphi(v, \tau) \equiv 0$, if $v < v_0(\tau)$, and for $v \geqslant v_0(\tau)$

$$\frac{\varphi(v,\tau)}{1-\tau} = \frac{\sqrt{1-\tau+\tau^2}}{\tau(v-v_0)} I_1\left(2\frac{v-v_0}{v_0}\sqrt{1-\tau+\tau^2}\right)\exp\left\{-(1-\tau+2\tau^2)\frac{v-v_0}{v_0\tau}\right\}. \quad (36)$$

If $\tau \ll 1$, then from formula (36) it follows, for $v > v_0$,

$$\varphi(v,\tau) \approx \frac{1}{v_0\tau}\exp\left\{-\frac{v-v_0}{v_0\tau}\right\}, \quad (37)$$

and when $\tau \ll 1-\tau$, for $v > v_0$, expression (21) is again obtained.

Let us note that, when considering the coagulation of drops in a strong ascending flow, interpolation formula (37) is not applicable to drops whose free-fall velocity is comparable with the velocity of the ascending flow, since the initial kinetic equation described only collisions of drops that had remained in the flow for equally long times.

In conclusion, the author considers it his duty to express gratitude to Corresponding Member of the Academy of Sciences of the USSR V. G. Levich for discussions.

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Note: Figure translations are in progress. See original paper for figures.