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# Mathematical Physics

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## Abstract

## Full Text

*Mathematical Physics*

KIM ZE PKHEN

# AN APPROXIMATE METHOD FOR SOLVING ONE TYPE OF SYSTEM OF LINEAR SINGULAR INTEGRAL EQUATIONS

*(Presented by Academician N. N. Bogolyubov on 4 February 1963)*

The work is devoted to an approximate method for solving a system of linear singular integral equations arising in the theory of dispersion relations.

An equation of the Chew-Low type for the process $\pi + N \to 2\pi + N$ has the form [1,2]

$$u(t) = -\frac{1}{\pi}\int_{-1}^{+1} \frac{v(\tau)}{\tau - t}\, d\tau,$$

$$\begin{aligned} v(t) = {} & [S_1(t) - N_1(-t)]\lambda + S_1(t)u(t) + S_2(t)v(t) - \\ & - N_1(-t)u(-t) + N_2(-t)v(-t), \end{aligned} \tag{1}$$

where $-1 \le t \le +1$; $u(v), v(t)$ are the unknown vector-functions; $S_1(t), S_2(t), N_1(t), N_2(t)$ are matrices belonging to $H(\alpha)$ and, for $t = \pm 1$, have zero of order $3/2$; $S_1(t) = S_2(t) = N_1(t) = N_2(t) = 0$ for $t \le 0$. We seek solutions of the system of equations (1) in the Hölder class $H(\alpha)$.

§ 1. Consider the system of equations (1) in $L_2$.

**Theorem 1.** If $\sqrt{n}\,(S_1 + S_2 + N_1 + N_2) \le q < 1$, where

$$S_\beta = \max_{i,j,t} |S_{\beta ij}(t)|, \qquad N_\beta = \max_{i,j,t} |N_{\beta ij}(t)|,$$

$$\beta = 1, 2; \qquad i, j = 1, 2, \ldots, s; \qquad -1 \le t \le +1,$$

then the system of equations (1) has a unique solution in the class $H(\alpha)$, and the successive approximations

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$$v_{m+1}(t) = [S_1(t) - N_1(-t)]\pi - S_1(t)\frac{1}{\pi}\int_{-1}^{+1}\frac{v_m(\tau)}{\tau - t}\,d\tau + S_2(t)v_m(t) +$$
$$+ N_1(-t)\frac{1}{\pi}\int_{-1}^{+1}\frac{v_m(\tau)}{\tau + t}\,d\tau + N_2(-t)v_m(-t), \qquad v_0(t) \in H(\alpha)$$

converge in norm in $L_2$.

Using the formula

$$\left\|\frac{1}{\pi}\int_{-1}^{+1}\frac{\varphi(\tau)}{\tau - t}\,d\tau\right\| \leq \|\varphi(t)\| \quad ({}^{3,4}),$$

it is easy to prove Theorem 1.

§ 2. Consider the following Riemann boundary-value problem.

**Problem 1.** Find a vector-function $\Phi(z) = (\Phi_1(z), \Phi_2(z), \dots, \Phi_s(z))$, analytic in the plane cut along $[-1, +1]$, continuous everywhere up to the boundary except perhaps for only a finite number of points on $[-1, +1]$, where integrable singularities are allowed, tending to zero at infinity and satisfying the boundary condition

$$\begin{aligned}(P(t) + i\sqrt{1-t^2}\,Q(t))\Phi^+(t) - (P(t) - i\sqrt{1-t^2}\,Q(t))\Phi^-(t) = \\ = 2if(t),\end{aligned} \qquad (2)$$

where $P(t), Q(t)$ are matrices whose elements are polynomials; $f(t)$ is a given

a vector-function satisfying the Hölder condition. $\Phi^{\pm}(t)$ are the limiting values of $\Phi(z)$, respectively for $z \to t + 0i,\ z \to t - 0i,\ t \in [-1, +1]$.

Problem 1 is easily solved by the method of Muskhelishvili ([5]).

**Theorem 2.** *If boundary-value problem 1 is solvable, then the solution is represented in the form*

$$\Phi(z) = X^{-1}(z)\left[\frac{1}{\pi}\int_{-1}^{+1}\frac{f(\tau)}{\tau - z}\,d\tau + R(z)\right], \qquad (3)$$

*where*

$$X(z) = P(z) + \sqrt{z^2 - 1}\,Q(z);$$

*$R(z)$ is a vector-function whose components are polynomials. $R(z)$ is determined from the condition of analyticity of $\Phi(z)$ in the plane with the cut $[-1, +1]$ and from its behavior at infinity. $R(z)$ may include arbitrary coefficients.*

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**Sec. 3.** Consider the system of singular integral equations

$$\overline{K}\varphi(t) \equiv P(t)\varphi(t) + \sqrt{1-t^2}\,Q(t)\frac{1}{\pi}\int_{-1}^{+1}\frac{\varphi(\tau)}{\tau - t}\,d\tau = f(t), \tag{4}$$

where $\varphi(t) \equiv (\varphi_1(t), \varphi_2(t), \dots, \varphi_s(t))$ is the sought vector-function; $P(t), Q(t), f(t)$ are the same as in problem 1.

Putting

$$\Phi(z) = \frac{1}{\pi}\int_{-1}^{+1}\frac{\varphi(\tau)}{\tau - z}\,d\tau,$$

the system of equations (4) can be reduced to problem 1 ([5]).

**Theorem 3.** *The system of equations (4), in the sense of solvability, is equivalent to problem 1. If problem 1 is solvable, then the solution of the system of equations (4) is given in the form*

$$\varphi(t) = \Gamma f(t) + B(t)R(t), \tag{5}$$

*where*

$$\Gamma f(t) \equiv A(t)f(t) + B(t)\frac{1}{\pi}\int_{-1}^{1}\frac{f(\tau)}{\tau - t}\,d\tau,$$

$$A(t) \equiv \frac{1}{2}\left([X^+(t)]^{-1} + [X^-(t)]^{-1}\right),$$

$$B(t) \equiv \frac{1}{2i}\left([X^+(t)]^{-1} - [X^-(t)]^{-1}\right).$$

**Sec. 4.** The system of equations (1) can be reduced to the following form

$$a(t)\varphi(t) + b(t)\frac{1}{\pi}\int_{-1}^{+1}\frac{\varphi(\tau)}{\tau - t} = f(t), \tag{6}$$

where

$$\varphi(t) = \left(\frac{v(t) + v(-t)}{2},\ \frac{v(t) - v(-t)}{2}\right),$$

$$a(t) = \begin{pmatrix} I - c_1(t) & -d_2(t) \\ -c_2(t) & I - d_1(t) \end{pmatrix},$$

$$b(t) = \begin{pmatrix} b_2(t) & a_1(t) \\ b_1(t) & a_2(t) \end{pmatrix},$$

$$f(t) = (a_1(t)\lambda,\ a_2(t)\lambda),$$

$I$ is the identity matrix,

$$a_1(t) = \frac{S_1(t) + S_1(-t) - N_1(t) - N_1(-t)}{2},$$

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$$a_2(t) = \frac{S_1(t) - S_1(-t) + N_1(t) - N_1(-t)}{2},$$
$$b_1(t) = \frac{S_1(t) + S_1(-t) + N_1(t) + N_1(-t)}{2}.$$

$$b_2(t) = \frac{S_1(t) - S_1(-t) - N_1(t) + N_1(-t)}{2},$$

$$c_1(t) = \frac{S_2(t) + S_2(-t) + N_2(t) + N_2(-t)}{2},$$

$$c_2(t) = \frac{S_2(t) - S_2(-t) - N_2(t) + N_2(-t)}{2},$$

$$d_1(t) = \frac{S_2(t) + S_2(-t)^2 - N_2(t) - N_2(-t)}{2},$$

$$d_2(t) = \frac{S_2(t) - S_2(-t) + N_2(t) - N_2(-t)}{2},$$

$$b_{ij}(\pm 1) = 0$$

and the order of the zero is 3/2.

It is then clear that any solution of system (1) is a solution of system (6), and conversely, from any solution of system (6), for $\varphi_1, \varphi_2, \dots, \varphi_s$ even and $\varphi_{s+1}, \varphi_{s+2}, \dots, \varphi_{2s}$ odd functions, a solution of system (1) is constructed by the formulas

$$v_1(t) = \varphi_1(t) + \varphi_{s+1}(t), \quad v_2(t) = \varphi_2(t) + \varphi_{s+2}(t), \dots, \quad v_s(t) = \varphi_s(t) + \varphi_{2s}(t).$$

In the case $\det \|a(t) + ib(t)\| \neq 0$ on $[-1; +1]$, the system of equations (6) is solved effectively.

We approximate $a(t), b(t)$ by polynomials (this is possible [6]). Suppose that

$$a(t) = P(t) + \xi(t), \qquad b(t) = \sqrt{1-t^2}\, Q(t) + \sqrt{1-t^2}\, \eta(t),$$

$$\max |\xi_{ij}(t)| < \varepsilon, \qquad \max |\eta_{ij}(t)| < \varepsilon,$$

where $\varepsilon > 0$ is a sufficiently small number. Equation (6) can be rewritten in the form

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$$\overline{K}\varphi(t) = f(t) - S\varphi(t), \tag{7}$$

where

$$S\varphi(t) \equiv \xi(t)\varphi(t) + \sqrt{1-t^2}\eta(t)\frac{1}{\pi}\int_{-1}^{+1}\frac{\varphi(\tau)}{\tau - t}\,d\tau,$$

$\overline{K}$ is the same operator as in Sec. 3.

By virtue of Theorem 3, if equation (7) has a solution, then it must be

$$\varphi(t) = \Gamma f(t) - \Gamma S\varphi(t) + B(t)R(t). \tag{8}$$

We choose $\varepsilon$ so that the condition $\|\Gamma S\| \leq q < 1$ is satisfied. Then, by Banach' s theorem, the solution of system (8) is given in the form

$$\varphi(t) = \sum_{l=0}^{\infty}(-\Gamma S)^l \Gamma f(t) + \sum_{l=0}^{\infty}(-\Gamma S)^l B(t)R(t). \tag{9}$$

The coefficients in $R(t)$ are undetermined.

Substituting (9) into (6), we can determine the coefficients $R(t)$.

Thus, by solving equation (6), we can find all solutions of the original equation.

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*Note: Figure translations are in progress. See original paper for figures.*

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