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Abstract

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MATHEMATICS

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ON THE GROWTH OF AN ENTIRE FUNCTION ALONG A RAY

(Presented by Academician M. A. Lavrent' ev on 9 V 1963)

Let $f(z)$ be an entire function of order ρ, and let $T(r)$ be its Nevanlinna characteristic:

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\varphi})| \, d\varphi.$$

In the present note the growth of $\ln |f(re^{i\varphi})|$ for fixed φ is compared with the growth of $T(r)$. Without loss of generality we shall assume that the fixed ray is $\varphi = 0$, i.e. the positive real axis. Pólya [(1)] constructed examples of entire functions of arbitrary order ρ, $0 \le \rho \le \infty$, for which

$$\limsup_{r\to\infty} \ln |f(r)|/T(r) = \infty.$$

We shall supplement this result by showing that the following estimate always holds:

$$\liminf_{r\to\infty} \frac{\ln |f(r)|}{T(r)} \le \begin{cases} \pi\rho \operatorname{cosec} \pi\rho, & 0 \le \rho \le 1/2, \\ \pi\rho, & \rho > 1/2. \end{cases} \tag{1}$$

The fact that estimate (1) cannot be improved follows from consideration of the well-known Mittag-Leffler function $E_\rho(z)$. Estimate (1) for $\rho = \infty$ is trivial, and for $0 \le \rho \le 1/2$ follows at once from an inequality obtained almost simultaneously by Valiron and Valiron [(2–4)]; the new result is estimate (1) for $1/2 < \rho < \infty$.

The proof of (1) (for $1/2 < \rho < \infty$) uses several lemmas.

Lemma 1. Let the functions $h(r)$ and $g(r)$ be continuous for $a \le r < \infty$, with $g(r) > 0$ for $a \le r < \infty$. Let $K(\eta, r) \ge 0$ be continuous for $0 \le \eta < \eta_0$ and $a \le r < \infty$, and let the integrals

$$\int_a^\infty h(r)K(\eta, r)\, dr, \qquad \int_a^\infty g(r)K(\eta, r)\, dr > 0$$

converge for $0 < \eta < \eta_0$, while

$$\int_a^\infty g(r)K(0, r)\, dr = \infty.$$

Then

$$\limsup_{r\to\infty} \frac{h(r)}{g(r)} \ge \limsup_{\eta\to 0+} \frac{\int_a^\infty h(r)K(\eta, r)\, dr}{\int_a^\infty g(r)K(\eta, r)\, dr},$$

$$\liminf_{r\to\infty} \frac{h(r)}{g(r)} \le \liminf_{\eta\to 0+} \frac{\int_a^\infty h(r)K(\eta, r)\, dr}{\int_a^\infty g(r)K(\eta, r)\, dr}.$$

A function $\rho(r)$, differentiable on $[1, \infty)$, is called a *refined order* [(5)] for an entire function $f(z)$ of order ρ, if the following conditions are satis-

the following conditions are fulfilled: a) $\lim_{r\to\infty} \rho(r) = \rho$; b) $\lim_{r\to\infty} r\rho'(r) \ln r = 0$; c) $\limsup_{r\to\infty} T(r)r^{-\rho(r)} = 1$.

Lemma 2. If $\rho(r)$ is a proximate order for an entire function $f(z)$ of order $\rho > 0$, then

$$\int_1^\infty T(r)r^{-\rho(r)-1}\, dr = \infty.$$

Lemma 3. For every entire function $f(z)$ of order ρ, $1/2 < \rho < \infty$, one can find a proximate order $\rho(r)$ such that for $1 \le R < \infty$ the following holds:

$$\limsup_{\eta\to 0+} \frac{\int_R^\infty r^{-\rho(r)-\eta-1} \ln^+ |f(r)|\, dr}{\int_R^\infty r^{-\rho(r)-\eta-1} T(r)\, dr} \le \pi\rho + \omega(R), \tag{2}$$

where $\lim_{R\to\infty}\omega(R)=0$.

From Lemma 1, putting $h(r)=\ln^+|f(r)|$, $g(r)=T(r)$, $K(\eta,r)=r^{-\rho(r)-\eta-1}$, $a=R$, we have

$$\liminf_{r\to\infty}\ln^+|f(r)|/T(r)\le\pi\rho+\omega(R),$$

whence, letting $R\to\infty$, we obtain (1).

In the case when $f(z)$ is an entire function of order $1/2<\rho<\infty$ of the **divergence class** ([5], p. 495), inequality (2) was obtained by Rauch ([6]) with $\rho(r)\equiv\rho$, $\omega(R)\equiv 0$*, and the results of Valiron ([8]) were used essentially. Our proof of Lemma 3 follows basically the same path, but the consideration of proximate order complicates all the calculations.

In the case $0\le\rho\le 1/2$, the inequality of Wiman and Valiron ([2–4]) implies an inequality stronger than (1):

$$\liminf_{r\to\infty}\ln M(r)/T(r)\le\pi\rho\,\mathrm{cosec}\,\pi\rho,\qquad M(r)=\max_{|z|=r}|f(z)|.$$

Pólya ([1]) conjectured that for $\rho>1/2$ the inequality

$$\liminf_{r\to\infty}\ln M(r)/T(r)\le\pi\rho,\tag{3}$$

is valid, but we have not been able to prove it. If, however, one additionally assumes that for some fixed φ_0 one has $\ln|f(re^{i\varphi_0})|\sim\ln M(r)$, then (3) follows from (1) for $\rho>1/2$ and, as a special case, a result of I. V. Ostrovskii ([9], p. 31).

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* In fact, A. Rauch asserts in ([6]) the existence of the limit on the left-hand side of (2) (under the assumptions he made), but this assertion is not justified by anything. In ([7]) Rauch attempts to prove an even stronger inequality than in ([6]), but his arguments contain gross errors.

Note: Figure translations are in progress. See original paper for figures.