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Abstract

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ON THE SPECTRAL FUNCTION OF A SELF-ADJOINT OPERATOR IN A SPACE WITH INDEFINITE METRIC

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Let Π_χ denote the Pontryagin space whose axioms are given in the article (^{1}). In contrast to (^{1,2}), here a different sign will be adopted for the scalar product (x, y) $(x, y \in \Pi_\chi)$, so that, as in the original article of L. S. Pontryagin (^{3}), in the corresponding coordinate system the scalar square (x, x) of a vector $x \in \Pi_\chi$ is expressed by the form $\sum \varepsilon_j |\xi_j|^2$ $(\varepsilon_j = \pm 1)$, containing exactly χ negative squares. The integer χ will henceforth be called the index of the Pontryagin space.

Self-adjoint operators in Π_χ will be called, for short, s.a. operators. By virtue of the well-known theorem (see (^{3,1})) on the separability of the root subspaces of an s.a. operator A corresponding to its non-real eigenvalues, in studying invariant subspaces of such an operator one may restrict oneself to the case where the spectrum $\sigma(A)$ of the operator is purely real. If $\sigma(A)$ consists of isolated real numbers, then, as is easy to see, to the operator A there corresponds an orthogonal projector-function E_λ $(-\infty < \lambda < \infty)$, having the property that the mutually orthogonal subspaces $E_\lambda \Pi_\chi$ and $(I - E_\lambda)\Pi_\chi$ contain all root subspaces corresponding respectively to eigenvalues $\leq \lambda$ and $> \lambda$.

The present note is devoted to the construction of as complete as possible an analogue of the indicated projector-function for an s.a. operator with arbitrary real spectrum. This construction makes it possible to reveal a number of structural properties of s.a. operators and to develop an operational calculus for them (which, however, is not presented here for lack of space).

1. Recall that in Π_χ a projector P $(\mathfrak{D}(P) = \Pi_\chi, P^2 = P)$ is called orthogonal if $x - Px \perp Px$, or, equivalently, if it is self-adjoint: $(Px, y) = (x, Py)$ $(x, y \in \Pi_\chi)$.

Definition. An operator-function E_λ, whose values are orthogonal projectors in Π_χ, is called a spectral function with critical points $\alpha_1, \alpha_2, \dots, \alpha_n$ $(-\infty < \alpha_1 < \alpha_2 < \dots < \alpha_n < \infty)$, if it is defined for all real λ distinct from $\alpha_1, \alpha_2, \dots, \alpha_n$, with the following conditions observed: 1) $E_\lambda E_\mu = E_\mu E_\lambda = E_{\min(\lambda,\mu)}$; 2) for any $x \in \Pi_\chi$ the function $(E_\lambda x, x)$ is nondecreasing on each interval containing none of the critical points $\alpha_1, \alpha_2, \dots, \alpha_n$, and, in the sense of strong convergence, $E_\lambda = E_{\lambda-0} (= \lim_{\mu\uparrow\lambda} E_\mu)$, $\lim_{\lambda\downarrow-\infty} E_\lambda = 0$, $\lim_{\lambda\uparrow\infty} E_\lambda = 1$; 3) for each α_j and every $\varepsilon > 0$ there is an element $x_j = x_j(\varepsilon)$ such that

$$(E_{\alpha_j-\varepsilon} x_j, x_j) > (E_{\alpha_j+\varepsilon} x_j, x_j).$$

The set of critical points $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ will be denoted by $s(E_\lambda)$.

For any closed interval $\Delta = [\lambda, \mu]$ with noncritical endpoints ($\lambda, \mu \notin s(A)$) we put $E(\Delta) = E_{\mu+0} - E_\lambda (= E_{\mu+0} - E_{\lambda-0})$. Similarly $E(\Delta)$ is defined for a half-open or open interval with noncritical endpoints.

If Δ is one of such intervals, then $E(\Delta)\Pi_\chi$ will be a certain Pontryagin or Hilbert space, depending on whether Δ contains at least one critical point or not. In the second case

An interval Δ will be called **regular**. If Δ contains exactly one point $\alpha \in s(E_\lambda)$ (in this case we shall write $\Delta \in D(\alpha)$), then $E(\Delta)\Pi_\chi$ is some Pontryagin space $\Pi_{\chi'}$, with index χ' completely determined by the point α. The number χ' will be called the **index** of the critical point α, and we shall write $\chi' = \chi(\alpha)$. Let $\Delta_j \in D(\alpha_j)$ ($j = 1, 2, \dots, n$) be a system of intervals whose closures are pairwise disjoint. Such a system corresponds to the decomposition of Π_χ into an orthogonal sum of subspaces

$$\Pi_\chi = \Pi_0 \oplus \Pi^{(1)} \oplus \Pi^{(2)} \oplus \cdots \oplus \Pi^{(n)}, \tag{1}$$

where Π_0 is some Hilbert subspace, and $\Pi^{(j)} = E(\Delta_j)\Pi_\chi$ are Pontryagin spaces of index $\chi_j = \chi(\alpha_j)$ ($j = 1, 2, \dots, n$), which in special cases may degenerate into finite-dimensional spaces with the corresponding metric*.

For $\alpha \in s(E_\lambda)$, denote by $\mathfrak{S}_\alpha$ the intersection of all subspaces $E(\Delta)\Pi_\chi$ corresponding to all possible $\Delta \in D(\alpha)$. It is easy to see that always $\dim \mathfrak{S}_\alpha > 0$.

Lemma. For a critical point $\alpha \in s(E_\lambda)$, the following assertions are equivalent: 1) the point α is a "regular" point for E_λ in the sense that there exist strong limits

$$E_{\alpha-0} = \lim_{\lambda\uparrow\alpha} E_\lambda, \qquad E_{\alpha+0} = \lim_{\lambda\downarrow\alpha} E_\lambda;$$

2) the intersection $\mathfrak{S}_\alpha$ and $\mathfrak{S}_\alpha^{\perp **}$ consists of zero alone; 3) $\mathfrak{S}_\alpha$ is a Pontryagin space of index $\chi(\alpha)$.

Assertion 2) means, in other words, that $\mathfrak{S}_\alpha$ contains no isotropic vectors. If the indicated assertions are valid for the point α, then $P_\alpha = E_{\alpha+0} - E_{\alpha-0}$ is an orthogonal projector and, naturally, $\mathfrak{S}_\alpha = P_\alpha \Pi_\chi$.

We shall agree to say that a critical point α of the spectral function E_λ has finite order if there exists an integer q (≥ 0) such that, for $\Delta \in D(\alpha)$, the integral

$$\int_\Delta (\lambda - \alpha)^{2q} \, dE_\lambda$$

converges strongly (as an improper integral with singular point α). The least nonnegative integer $q = q(\alpha)$ for which this integral converges is called the **order** of the critical point α. If the order $q(\alpha) = 0$, then this means that the point α is regular.

2. By the fundamental theorem of L. S. Pontryagin (^{3}), every s.s. operator A with real spectrum always has at least one χ-dimensional nonpositive subspace $\mathcal{L}_A \in \mathfrak{D}(A)$, invariant with respect to A. Choose $\mathcal{L}_A$ so that the minimal polynomial

   $$\mathcal{P}_A(\lambda) = (\lambda - \alpha_1)^{r_1} \cdots (\lambda - \alpha_n)^{r_n}$$

   of the operator A in $\mathcal{L}_A$ ($\mathcal{P}_A(A)\mathcal{L}_A = \{0\}$) has the smallest possible degree. The real numbers $\alpha_1, \dots, \alpha_n$ will be called the **critical numbers** of the operator A. The set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, completely determined by the operator A (^{2}), will be denoted by $s(A)$.

Theorem 1. To every s.s. operator A (with real spectrum) acting in Π_χ, there corresponds a unique spectral function E_λ with critical points (called the "proper" one) having the properties: 1) $s(E_\lambda) = s(A)$; 2) for any finite interval Δ with noncritical endpoints, $E(\Delta)\Pi_\chi \subset \mathfrak{D}(A)$, $E(\Delta)Ax = AE(\Delta)x$ for $x \in \mathfrak{D}(A)$, and the spectrum of the operator A in $E(\Delta)\Pi_\chi$ lies in $\overline{\Delta}$ (the closure of Δ); 3) for any regular interval Δ:

$$AE(\Delta)x = \int_\Delta \lambda \, dE_\lambda x,$$

where the integral converges strongly.

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[*] For finite-dimensional $\Pi^{(j)}$ it may turn out that in it $(x, x) < 0$ ($x \neq 0$). In what follows, cases of degeneration of a Pontryagin or Hilbert space into finite-dimensional ones are not stipulated separately.

[**] If $\mathfrak{M} \subset \Pi_\chi$, then by $\mathfrak{M}^\perp$ is denoted the set of all $x \in \Pi_\chi$ orthogonal to $\mathfrak{M}$: $(x, y) = 0,\ y \in \mathfrak{M}$.

The eigenspectral function E_λ also has the following property:

4) for any finite interval Δ with noncritical endpoints,

$$\mathcal{P}_A^2(A)E(\Delta)x = \int_\Delta \mathcal{P}_A^2(\lambda)\, dE_\lambda x \quad (x \in \Pi_\chi), \tag{2}$$

where the integral converges strongly as an improper integral with singularities $\alpha_j \in \Delta$.

Let us give some explanations for the proof of the theorem. In order not to complicate the argument with inessential details, suppose that A is a bounded operator ($\mathfrak{D}(A) = \Pi_\chi$). By means of the polynomial $\mathcal{P}_A(\lambda)$ we form the polynomial in A and ζ: $Q(A,\zeta) = (A-\zeta I)^{-1}(\mathcal{P}_A^2(A) - \mathcal{P}_A^2(\zeta)I)$. Then for $R_\zeta = (A-\zeta I)^{-1}$ $(\zeta \notin \sigma(A))$ and arbitrary $x, y \in \Pi_\chi$ we shall have

$$(R_\zeta x, y) = \frac{1}{\mathcal{P}_A^2(\zeta)}(R_\zeta \mathcal{P}_A(A)x, \mathcal{P}_A(A)y) - \frac{1}{\mathcal{P}_A^2(\zeta)}(Q(A,\zeta)x, y). \tag{3}$$

We note that the subtracted term on the right-hand side is a proper rational function of ζ with poles in $s(A)$. The set of values of the operator $\mathcal{P}_A(A)$ belongs to $\mathfrak{L}_A^\perp$. Since the subspace $\mathfrak{L}_A^\perp$ is nonnegative and invariant with respect to A, for $\operatorname{Im}\zeta \neq 0$ we shall have

$$(R_\zeta \mathcal{P}_A(A)x, \mathcal{P}_A(A)y) = \int_{+\infty}^{\infty} \frac{d\sigma_t(x,y)}{t-\zeta} \quad (x, y \in \Pi_\chi), \tag{4}$$

where $\sigma_t(x,y)$, for arbitrary $x, y \in \Pi_\chi$, is a function of bounded variation of $t \in (-\infty, \infty)$, and is nondecreasing when $x = y$.

The use of representation (4) in equality (3) makes it possible to prove the existence of such an operator-function F_λ $(-\infty < \lambda < \infty,\ \lambda \notin s(A))$ that, for any real $\lambda, \mu \notin s(A)$ $(\lambda < \mu)$,

$$(F_\mu - F_\lambda)x = -\frac{1}{2\pi i} \oint_{\Gamma'(\lambda,\mu)} R_\zeta x \, d\zeta,$$

where $\Gamma(\lambda,\mu)$ is an arbitrary positively oriented smooth contour intersecting the real axis at the points λ and μ at right angles, and the prime on the integral sign means that the principal value of the integral is taken (with singular points λ, μ), existing in the sense of strong convergence.

It turns out that the function F_λ differs only inessentially from the desired spectral function E_λ, namely: $E_\lambda = F_{\lambda-0}$, and $F_\lambda = \frac{1}{2}(E_{\lambda+0} + E_{\lambda-0})$.

3. After Theorem 1, for a point $\alpha \in s(A)$ the notions of the index $\varkappa(\alpha)$, the order $q(\alpha)$, and the subspace $\mathfrak{S}_\alpha$, which we identify with the corresponding notions for the point α as a critical point of the eigenspectral function E_λ of the self-adjoint operator A, acquire meaning.

We note that to the orthogonal decomposition (1), generated by this spectral function E_λ, there will correspond a decomposition of the operator A into a direct sum: $A = A_0 \oplus A^{(1)} \oplus \cdots \oplus A^{(n)}$, where A_0 and $A^{(j)}$ are self-adjoint operators induced in the invariant subspaces Π_0 and $\Pi^{(j)}$ $(j = 1, 2, \ldots, n)$ by the operator A, and each operator $A^{(j)}$ will be bounded and will have the single critical number α_j, and consequently for it $\mathcal{P}_{A^{(j)}}(\lambda) = (\lambda - \alpha_j)^{r_j}$ $(j = 1, 2, \ldots, n)$. Hence it is clear that $r_j \leq \varkappa(\alpha_j)$. On the other hand, from (2) it follows that $q(\alpha_j) \leq r_j$. Thus $q(\alpha_j) \leq \varkappa(\alpha_j)$ $(j = 1, 2, \ldots, n)$.

Denote by $\mathfrak{J}_\alpha$ ($\alpha \in s(A)$) the isotropic lineal of the subspace $\mathfrak{S}_\alpha$: $\mathfrak{J}_\alpha = \mathfrak{S}_\alpha \cap \mathfrak{S}_\alpha^\perp$, and put $p(\alpha) = \dim \mathfrak{J}_\alpha$.

Theorem 2. The subspace $\mathfrak{S}_\alpha$ coincides with the root lineal of the self-adjoint operator A, corresponding to its eigenvalue $\alpha \in s(A)$*, and

$$(A - \alpha I)^{\nu(\alpha)} \mathfrak{S}_\alpha = \{0\}, \quad \text{where} \quad \nu(\alpha) = q(\alpha) + 2(\varkappa(\alpha) - p(\alpha)) + 1.$$

* That is, with the set of all those $x \in \Pi_\chi$ for which there exists a natural number $k = k(x)$ such that $x \in \mathfrak{D}(A^k)$ and $(A - \alpha I)^k x = 0$.

The second assertion of the theorem means that the length of any Jordan chain of the operator A in $\mathfrak{S}_a$ does not exceed the number $\nu(a)$. In deriving this assertion one uses an important characteristic of the number $q(a)$ in its own right, which makes it possible to determine it for $a \in s(A)$ without invoking the notion of the proper spectral function of the self-adjoint operator A.

Theorem 3. *The order $q(a)$ of a point $a \in s(A)$ coincides with the least natural number q for which*

$$(A - aI)^q \mathfrak{S}_a = \{0\}.$$

Consequently, $q(a) \leq p(a)$ and $\nu(a) \leq 2\varkappa(a) - q(a) + 1$.

4. Suppose that in $\Pi_\varkappa$ a certain spectral function E_λ is given with critical points a_j ($j = 1, 2, \dots, n$). From the preceding it is clear that there will not always exist in $\Pi_\varkappa$ at least one self-adjoint operator A with real spectrum for which E_λ will be its proper spectral function. Indeed, for this it is necessary that $q(a_j) \leq \varkappa(a_j)$ ($j = 1, 2, \dots, n$). It can be shown that these conditions, generally speaking, are not sufficient. However, they are sufficient if all $\varkappa(a_j) = 1$ ($j = 1, 2, \dots, n$), in particular if $\varkappa = 1$. Moreover, one may assert that the spectral function E_λ will be proper for some self-adjoint operator A with real spectrum whenever $q(a_j) \leq 1$ ($j = 1, 2, \dots, n$). In order to simplify the formulations of the corresponding results we shall give them for the case $n = 1$.

Theorem 4. *Let E_λ be a spectral function in $\Pi_\varkappa$ with the unique critical point a of order $q(a) = 1$. Define, by the equality*

$$A_0 x = ax + \int_{-\infty}^{\infty} (\lambda - a)\, dE_\lambda x \tag{5}$$

the operator A_0 on the set $\mathfrak{D}_0$, consisting of all those $x \in \mathfrak{S}_a^\perp$ for which the integral (5) converges strongly. Then $A_0 \mathfrak{D}_0 \subset \mathfrak{S}_a^\perp$, and $\mathfrak{D}_0$ is dense in $\mathfrak{S}_a^\perp$ and, moreover, contains every subspace $E(\Delta)\mathfrak{S}_a^\perp$ (Δ an arbitrary finite interval).

The operator A_0 will admit infinitely many self-adjoint extensions A in $\Pi_\varkappa$ having $\mathfrak{S}_a$ as the root subspace corresponding to the eigenvalue a. All self-adjoint operators A in $\Pi_\varkappa$ obtained in this way (and only they) will have E_λ as their proper spectral function.

In the case $q(a) = 0$ Theorem 4 is also true,* but in this case there exists an orthogonal projector $P_a = E_{a+0} - E_{a-0}$, and everything simplifies. The space $\mathfrak{S}_a^{\perp} = (I - P_a)\Pi_{\varkappa}$ will be Hilbert, the operator A_0 will be self-adjoint in $\mathfrak{S}_a^{\perp}$, and every self-adjoint extension A in $\Pi_{\varkappa}$ of the required type will be obtained as the direct sum $A = A_1 \oplus A_0$, where A_1 is any self-adjoint operator in the Pontryagin space $P_a\Pi_{\varkappa}$ with the unique point of the spectrum—the eigenvalue a (i.e., the operator $A_1 - aI$ is nilpotent in $\mathfrak{S}_a$).

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* Except that, in the case when the subspace $\mathfrak{S}_a$ is negative, there will already exist a unique self-adjoint extension A of the operator A_0 of the required type (in this case $Ax = aP_a x + A_0(I - P_a)x,\ x \in \mathfrak{D}(A) = \mathfrak{D}_0 + \mathfrak{S}_a$).

Note: Figure translations are in progress. See original paper for figures.

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