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Abstract

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On the Complex Structures of One Class of Kähler-Rashevskii Spaces

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We shall call a Riemannian space V_n with metric tensor g_{ij} a B-space if it admits a covariantly constant affinor γ_i^j such that the tensor $\gamma_{ij} = \gamma_i^k g_{kj}$ is symmetric. Spaces of this type were studied in papers ([1–3]). Since the geometric properties of a B-space depend essentially on the algebraic structure of the affinor γ_i^j, we shall call the characteristic (([4]), p. 182) of its matrix the characteristic of the B-space.

Using A. P. Shirokov' s theorem ([5]) on the existence of a holonomic coordinate system in which the matrix of a covariantly constant affinor of a torsion-free affine-connection space has constant components, G. I. Kruchkovich and A. S. Solodovnikov ([3]) showed that only B-spaces of characteristics $[(m_1, m_2, m_3, \dots, \dots, m_r)]$—with one real eigenvalue—and $[(n_1, n_2, n_3, \dots, \dots, n_r), (n_1, n_2, n_3, \dots, n_r)]$—with two complex conjugate eigenvalues—are irreducible. The present paper aims to study one important class of irreducible B-spaces with a real eigenvalue λ, which may be taken to be zero, by considering the new covariantly constant affinor $\gamma_i^j - \lambda\delta_i^j$, as will be assumed in what follows.

Since the tensors

$$\overset{s}{\gamma}{}_i^j = \overbrace{\gamma_i^p \gamma_p^q \cdots \gamma_q^j}^{s \text{ times}}$$

are covariantly constant, in the coordinate system of A. P. Shirokov we have

$$\Gamma_{km}^j \overset{s}{\gamma}{}_i^m = \Gamma_{ki}^l \overset{s}{\gamma}{}_l^j, \tag{1}$$

whence it follows that the components of the symmetric tensors $\overset{p}{\gamma}_{ij} = g_{ik}\overset{p}{\gamma}{}_j^k$ satisfy the system of differential equations

$$\overset{q}{\gamma}{}_k^m\, \partial_m \overset{p}{\gamma}_{ij} = \partial_k \overset{p+q}{\gamma}_{ij}, \tag{2}$$

which, in the four-dimensional case, was first obtained by A. P. Norden ([1]). This system has the simplest geometric meaning when in the characteristic of the B-space $m_1 = m_2 = \dots = m_r = m$, and consequently, in a certain holonomic coordinate system, which we shall call canonical, the matrix of the affinor γ_i^j will have on its main diagonal r Jordan blocks of dimension m. In consequence of the relations $g_{ik}\gamma_j^k = g_{jl}\gamma_i^l$, the matrix of the metric tensor in this coordinate system decomposes into blocks of the form

$$B_{\alpha\beta} = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_m \\ b_2 & b_3 & \cdots & b_m & 0 \\ b_3 & \cdots & b_m & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_m & 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad \alpha, \beta = 1, 2, \dots, r. \tag{3}$$

The matrices $\overset{1}{\gamma}_{ij}, \overset{2}{\gamma}_{ij}, \dots, \overset{m-1}{\gamma}_{ij}$, having an analogous structure, are obtained from (3) by raising (when contracting with $\overset{s}{\gamma}_i$) each element in all cells by s places upward and replacing the vacated places by zeros.

Let us establish the meaning of the differential equations (2). For $k = 1, 2, \dots, m$ we arrive at the system

$$\begin{aligned} &\partial_2 g_{ij} = \partial_1 \overset{1}{\gamma}_{ij}, \quad \partial_3 g_{ij} = \partial_2 \overset{1}{\gamma}_{ij} = \partial_1 \overset{2}{\gamma}_{ij}, \quad \partial_4 g_{ij} = \partial_3 \overset{1}{\gamma}_{ij} = \partial_2 \overset{2}{\gamma}_{ij} = \partial_1 \overset{3}{\gamma}_{ij}, \ \dots \\ &\dots, \partial_m g_{ij} = \partial_{m-1} \overset{1}{\gamma}_{ij} = \partial_{m-2} \overset{2}{\gamma}_{ij} = \dots = \partial_1 \overset{m-1}{\gamma}_{ij}, \quad \partial_m \overset{1}{\gamma}_{ij} = \partial_{m-1} \overset{2}{\gamma}_{ij} = \partial_{m-2} \overset{3}{\gamma}_{ij} = \dots \\ &\dots = \partial_2 \overset{m-1}{\gamma}_{ij} = 0, \quad \partial_m \overset{2}{\gamma}_{ij} = \partial_{m-1} \overset{3}{\gamma}_{ij} = \dots = \partial_3 \overset{m-1}{\gamma}_{ij} = 0, \ \dots \\ &\dots, \quad \partial_m \overset{m-2}{\gamma}_{ij} = \partial_{m-1} \overset{m-1}{\gamma}_{ij} = 0, \quad \partial_m \overset{m-1}{\gamma}_{ij} = 0. \end{aligned} \tag{4}$$

Replacing here the differentiation indices $1, 2, 3, \dots, m$ by $ms+1$, $ms+2$, $ms+3, \dots, m(s+1)$, respectively, where $s = 1, 2, \dots, r-1$, we obtain another $r-1$ systems analogous to (4). Introducing the imaginary unit ε, satisfying the condition $\varepsilon^m = 0$, we assert that conditions (4) are necessary and sufficient for the components of the tensor

$$G_{ij} = \overset{m-1}{\gamma}_{ij} + \varepsilon \overset{m-2}{\gamma}_{ij} + \varepsilon^2 \overset{m-3}{\gamma}_{ij} + \dots + \varepsilon^{m-2} \overset{1}{\gamma}_{ij} + \varepsilon^{m-1} g_{ij} \tag{5}$$

to be analytic functions of r complex coordinates

$$U^s = u^{m(s-1)+1} + \varepsilon u^{m(s-1)+2} + \varepsilon^2 u^{m(s-1)+3} + \dots + \varepsilon^{m-1} u^{ms}, \quad s = 1, 2, \dots, r.$$

The matrix of the tensor G_{ij} has a cellular structure; moreover, in consequence of the described structure of the matrices $g_{ij}, \overset{1}{\gamma}_{ij}, \dots, \overset{m-1}{\gamma}_{ij}$, the cells $G_{\alpha\beta}$ have the form

$$\begin{pmatrix} G_{\alpha\beta} & \varepsilon G_{\alpha\beta} & \varepsilon^2 G_{\alpha\beta} & \cdots & \cdots & \varepsilon^{m-1} G_{\alpha\beta} \\ \varepsilon G_{\alpha\beta} & \varepsilon^2 G_{\alpha\beta} & \cdots & \cdots & \varepsilon^{m-1} G_{\alpha\beta} & 0 \\ \varepsilon^2 G_{\alpha\beta} & \cdots & \cdots & \varepsilon^{m-1} G_{\alpha\beta} & 0 & 0 \\ . & \cdots & . & . & . & . \\ . & \cdots & . & . & . & . \\ \varepsilon^{m-1} G_{\alpha\beta} & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Observing that each of them is determined by its left upper corner element, we come to the conclusion that the B-space of characteristic

$$\Big[\Big(\overbrace{m, m, m, \dots, m}^{r}\Big)\Big]$$

admits a mapping onto an r-dimensional complex analytic Riemannian space with metric tensor $G_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \dots, r)$. For $m = 3$ these results were first obtained by A. P. Shirokov (6).

In what follows we shall dwell in detail on the B-space of characteristic $[(m, m)]$, since it is mapped onto the two-dimensional complex space V_2, and the properties of the latter determine a number of interesting properties of this B-space.* In particular, in V_2 one can introduce an isothermal coordinate system in which the complex metric has the form

$$(G_{\alpha\beta}) = \begin{pmatrix} B & 0 \\ 0 & -\sigma B \end{pmatrix} \quad (\alpha, \beta = 1, 2),$$

where $\sigma = \pm 1$ determines the signature of this complex V_2. Then for the metric of the original B-space we obtain a matrix of the same structure, where B is an m-dimensional cell of the form (3).

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* The study of the B-space of characteristic $[m]$ is of no interest, since it is Euclidean ((3), note on p. 155).

Consider the discriminant bivector $E_{\alpha\beta}$ of the complex V_2 and define its essential component by means of the equality

$$E_{12}^2 = -\sigma \left(G_{11} G_{22} - G_{12}^2\right).$$

Then in isothermal coordinates $E_{12} = B$, or, in more detail,

$$e_{12}^{m-1} + \varepsilon e_{12}^{m-2} + \varepsilon^2 e_{12}^{m-3} + \cdots + \varepsilon^{m-1} e_{12}^0 = \gamma_{11}^0 + \varepsilon \gamma_{11}^{m-1} + \varepsilon^2 \gamma_{11}^{m-2} + \cdots + \varepsilon^{m-1} \gamma_{11}^{m-3},$$

whence, by comparison, we obtain $e_{12}^p = \gamma_{11}^p$ $(p = 0, 1, 2, \dots, m-1;\ \gamma_{ij}^0 = g_{ij})$.

Returning to the B-space, we shall have m bivectors e_{ij}^p, whose matrices in the isothermal coordinate system have the form

$$\left(e_{ij}^p\right) = \begin{pmatrix} 0 & B_p \\ -B_p & 0 \end{pmatrix}, \tag{6}$$

where B_p is an entry of the matrix γ_{ij}^p, and which, as a consequence, satisfy the relations

$$\gamma_i^{pk} e_{kj}^q = e_{ij}^{p+q}. \tag{7}$$

Let us now transform the isothermal coordinates u^i according to the formulas

$$x^\alpha = \frac{1}{\sqrt{2}}\left(u^\alpha + e u^{\alpha+m}\right), \qquad \bar{x}^\alpha = \frac{1}{\sqrt{2}}\left(u^\alpha - e u^{\alpha+m}\right), \qquad \alpha = 1, 2, \dots, m, \tag{8}$$

where e is the imaginary unit, defined by the condition $e^2 = \sigma$. In the coordinate system $(x^\alpha, \bar{x}^\alpha)$, the matrix g_{ij} takes the form

$$\left(g_{ij}\right) = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}. \tag{9}$$

The structure of the matrices γ_{ij}^p will be analogous, and, consequently, the transformation (8) is equivalent to passing to isotropic coordinates of the complex V_2.

Writing the systems of equations (4) in the new variables, we obtain that, by virtue of them, the Kähler-Rashevskii conditions are satisfied:

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{x}^\gamma} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{x}^\beta}, \qquad \frac{\partial g_{\bar{\alpha}\beta}}{\partial x^\gamma} = \frac{\partial g_{\bar{\alpha}\gamma}}{\partial x^\beta}, \tag{10}$$

as a result of which our space, for $\sigma = -1$, will be a Kähler space ([7]), and for $\sigma = 1$, the so-called stratified space of P. K. Rashevskii ([8]), whose complex interpretation by means of double numbers was given by B. A. Rozenfeld ([9]). The connection of B-spaces with Kähler-Rashevskii manifolds was first pointed out by A. P. Norden ([1]).

Since the remaining tensors γ_{ij}^p have a structure similar to that of the tensor g_{ij}, and the conditions (10) are satisfied for them as well, they are also Kähler-Rashevskii metrics, which, however, will be degenerate; moreover, the degree of degeneracy increases together with p.

Transforming the bivectors (6) by means of (8), we conclude that

$$\left(e^p_{ij}\right) = \begin{pmatrix} 0 & -\sigma e \, \gamma^p_{\alpha\bar\beta} \\ \sigma e \, \gamma^p_{\alpha\bar\beta} & 0 \end{pmatrix},$$

whence it follows that each of the bivectors e^p_{ij} is covariantly constant and corresponds to "its own" Kähler metric (cf. ([7]), pp. 119–120).

As a consequence of (10), there exist real functions $\overset{p}{\Gamma}(x^\alpha, \bar x^\alpha)$, which we shall call Kern functions, such that

$$\overset{p}{\gamma}_{\alpha\bar\beta} = \partial^2 \overset{p}{\Gamma} / \partial x^\alpha \partial \bar x^\beta, \qquad p = 0, 1, 2, \dots, m-1. \tag{11}$$

Comparing (9) and the analogous matrices of the remaining metrics, we write down the complete set of relations between their components; whence, by virtue of (11), we obtain that the Kern functions satisfy the system of differential equations

$$\frac{\partial \overset{r-s}{\Gamma}}{\partial x^{m-r}} - \frac{\partial \overset{r-s-1}{\Gamma}}{\partial x^{m-r+1}} = f^{r-s}_{m-r}(x^\alpha), \qquad r = s+1, \dots, m-1; \quad s = 0, 1, 2, \dots, m-2;$$

$$\frac{\partial \overset{q}{\Gamma}}{\partial x^{m-t}} = \varphi^q_t(x^\alpha), \qquad t \le q-1; \quad q = 1, 2, \dots, m-1. \tag{12}$$

As a consequence of (11), the metrics $\overset{p}{\gamma}_{ij}$ do not change if the Kern functions are replaced by new ones according to the formulas

$$\overset{p}{\Gamma}{}'(x^\alpha, \bar x^\alpha) = \overset{p}{\Gamma}(x^\alpha, \bar x^\alpha) - \overset{p}{\Gamma}(x^\alpha, \bar c^\alpha) - \overset{p}{\Gamma}(\bar x^\alpha, c^\alpha), \tag{13}$$

where $c^\alpha = \text{const}$. Solving (13) with respect to $\overset{p}{\Gamma}(x^\alpha, \bar x^\alpha)$, and then substituting them into equations (12), which are satisfied for arbitrary $\bar x^\alpha$, we obtain that the new Kern functions $\overset{p}{\Gamma}{}'$ satisfy the system of equations (12) without right-hand sides, and, owing to the reality of $\overset{p}{\Gamma}{}'$, also the system obtained from it by overlining all x^α. After passing, by means of (8), to the coordinates u^i, we arrive at systems analogous to (4), which express the fact that the combination

$$\mathcal{G} = \overset{m-1}{\Gamma} + \varepsilon \overset{m-2}{\Gamma} + \varepsilon^2 \overset{m-3}{\Gamma} + \dots + \varepsilon^{m-2} \overset{1}{\Gamma} + \varepsilon^{m-1} G \qquad \left(G = \overset{0}{\Gamma}\right) \tag{14}$$

is an analytic function of the complex coordinates $U^1 = u^1 + \varepsilon u^2 + \varepsilon^2 u^3 + \dots + \varepsilon^{m-1} u^m$, $U^2 = u^{m+1} + \varepsilon u^{m+2} + \varepsilon^2 u^{m+3} + \dots + \varepsilon^{m-1} u^{2m}$.

Conversely, if such a function is given and the algebra of complex or dual numbers over which the Kähler space is constructed is specified, then, passing to the variables $x^\alpha, \bar{x}^\alpha$, one can construct on the functions $\overset{p}{\Gamma}$ Kähler metrics that will have a special structure of type (9). Referring this space to the coordinates u^i, we arrive at a B-space of characteristic $[(m, m)]$.

Summarizing what has been said, one may assert that the specification of a B-space of characteristic $[(m, m)]$ is equivalent to the specification of a function (14) of two complex variables U^1, U^2 and of an algebra of complex or dual numbers.

The present work generalizes results obtained by the author for the four-dimensional case [10].

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Note: Figure translations are in progress. See original paper for figures.