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Abstract

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OPTIMAL CONTROL IN A BANACH SPACE

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In a Banach space an equation and initial conditions are given:

$$\frac{dx(t)}{dt} = f(x(t), u(t)), \qquad x(a) = x_0 \quad (a \leq t \leq b), \tag{1}$$

where $u(t)$—the "control"—is a measurable function with values in a given set U of some topological space. This control is to be chosen so as to minimize a functional of the form

$$\int_0^b f^0(x(t), u(t))\, dt,$$

and certain conditions are imposed on $x(b)$ (for example, $x(b) = x_1$, or $x(b) \in S$, where S is a given manifold, or $x(b) \in T$, where T is a convex body). For a finite-dimensional space such problems were studied in (^{1}). Under certain restrictions on $f(x, u)$ we obtain necessary conditions for optimality of the control in the form of Pontryagin' s maximum principle.

1. Let $x(b) = x_1$ be a given point, but the time instant b is not fixed. It is assumed that problem (1), for fixed $u(t)$, has a unique solution $x(t)$, which lies in B_1 for almost all t in (a, b), and $f(x, u) \in B_2$ if $x \in B_1$ and $u \in U$ ($B_1 \subset B_2$); $\varphi(x, u)$ and $\partial\varphi(x, u)/\partial x$ are continuous in x and u. Denote

$$x^0(t) = \int_a^t f^0(x(\xi), u(\xi))\, d\xi, \quad y = (x^0; x), \quad \varphi = (f^0; f),$$

$$A_1 = R^1 \times B_1, \quad A_2 = R^1 \times B_2.$$

If, for $t_1 \leq t \leq t_2$, $u(t)$ does not depend on t, then $x(t)$—the solution of (1)—is continuous on this interval in the norm of B_1.

i_1) Let $\tilde{u}_1(t) = u(t)$ for $a < t < \tau - \varepsilon$ and $\tau < t < b$, $\tilde{u}_1(t) = v \in U$ if $\tau - \varepsilon < t < \tau$; $\tilde{u}_2(t) = u(t)$ for $a < t < \tau - \varepsilon$, $\tilde{u}_2(t) = u(t + \varepsilon)$ when $\tau - \varepsilon < t < b - \varepsilon$; $\tilde{u}_3(t) = u(t)$ for $a < t < \tau$, $\tilde{u}_3(t) = u(\tau)$ for $\tau < t < \tau + \varepsilon$, $\tilde{u}_3(t) = u(t - \varepsilon)$ if $\tau + \varepsilon < t < b + \varepsilon$.

Let $\tilde{x}_i(t)$ be the solution of (1) for $u(t) = \tilde{u}_i(t)$. Put $\theta_1(t,\tau) \equiv 0$; $\theta_2(t,\tau) = 0$ for $t < \tau$, $\theta_2(t,\tau) = -1$ for $t \geq \tau$; $\theta_3(t,\tau) = -\theta_2(t,\tau)$.

There exists a set M of full measure on (a,b) of points τ, for which there exists

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}[\tilde{y}_i(t + \varepsilon\theta_i(t,\tau)) - y_i(t)] = z_i(t) \in A_2.$$

Moreover,

$$\lim_{\varepsilon \to 0} \|\tilde{x}_i(b + \varepsilon\theta_i(b,\tau)) - x(b)\|_{B_1} = 0 \quad (i = 1,2,3).$$

i_2) Let $a^* \in A_2^*$ (i.e., a^* is a linear continuous functional on A_2; the value of a^* at $y \in A_2$ is denoted by (a^*, y)). There exists a function $\psi(t)$ such that $\psi(b) = a^*$; for almost all t in (a,b), $\psi(t) \in A_2^*$; $(\psi(t), z_i(t)) = \text{const}$ $(i = 1,2,3;\ \tau \leq t \leq b)$, and, if $\psi(t) = (\psi_0(t), \psi_1(t))$, where $\psi_1(t) \in B_2^*$, then $\psi_0(t) = \text{const}$.

i_3) The reserve of controls U is sufficiently large. More precisely, there exist $\rho > 0$ and a set $\omega \subset U$ such that, for x in the ball $\|x - x_1\|_{B_1} \leq \rho$, the set $\{f(x,u)\}_{u \in \omega}$ is open and homeomorphic to ω.

Theorem 1. *If conditions i_1), i_2), i_3) are fulfilled, then for every optimal control $u(t)$ there exists $a^* \in A_2^*$ $(a^* \neq 0)$ such that every function $\psi(t)$ from i_2) has the property that the function*

$$H(\psi(t), x(t), u) = (\psi(t), \varphi(x(t), u)) \tag{2}$$

of the variable $u \in U$ almost everywhere on the interval $a \leq t \leq b$ attains a maximum at the point $u = u(t)$. (Here $x(t)$ is the solution of (1) corresponding to $u(t)$.) Moreover, $H(\psi(t), x(t), u(t)) \equiv 0$, $\psi_0 \leq 0$.

Remark 1. If $f(x,u)$ is a bounded operator, $B_1 = B_2$, then $z(t)$ and $\psi(t)$ satisfy the equations

$$\frac{dz(t)}{dt} = \frac{\partial\varphi(x(t),u(t))}{\partial x} z(t), \qquad \frac{d\psi(t)}{dt} = -\left(\frac{\partial\varphi(x(t),u(t))}{\partial x}\right)^* \psi(t).$$

If, however, f is an unbounded operator, $z(t)$ and $\psi(t)$ satisfy these equations only in a certain generalized sense.

Remark 2. The essential nature of requirement i_3 in the case of an infinite-dimensional space is shown by the following example.

If in l_2 the problem is posed: minimize the time $b - a$ (i.e. $f^0 \equiv 1$) under the conditions

$$\frac{dx}{dt} = u, \qquad x_n(a) = 0, \qquad x_n(b) = \frac{1}{n},$$

$$U = \left\{ u: \ |u_n| \leq \frac{1}{n} + \frac{1}{n^2} \right\} \qquad (n = 1, 2, \dots),$$

then, as is easy to see, the control $u(t) = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right)$ is optimal, and there does not exist a vector $\psi(t)$ satisfying the conditions of the maximum principle. If, however, x and u are regarded as elements of a space in whose topology the set U has an interior point, such a vector $\psi(t)$ always exists.

2. Let S be a smooth manifold in B_1, and suppose that in the optimal-control problem it is required that $x(b) \in S$ (the time b is not fixed).

j_1) Let T_1 be the tangent manifold to S at the point $x(b)$, and $T_2 = B_2/T_1$. Denote by y^π the class in T_2 in which $y \in B_2$ lies. There exist $\rho > 0$ and a set $\omega \subset U$ such that, for every x in the ball $\|x - x(b)\|_{B_1} \leq \rho$, the set $\{f^\pi(x, u)\}_{u \in \omega}$ is open in T_2 and homeomorphic to ω.

Theorem 2. If conditions i_1), i_2), j_1) are fulfilled, then for every optimal control $u(t)$ there exists $a^* \in A_2^*$ $(a^* \neq 0)$, satisfying the transversality conditions (i.e. $(a^*, y) = 0$ for $y \in T_1$) and such that, whatever the function $\psi(t)$ from condition i_2) may be, the function (2) attains a maximum with respect to u at the point $u = u(t)$. Moreover, $H(\psi(t), x(t), u(t)) \equiv 0, \ \psi_0 \leq 0$.

Remark 1. The theorem is also valid in the case when $x(a)$ is not fixed, but merely belongs to a given manifold S_0. In this case $\psi(t)$ satisfies the transversality conditions also for $t = a$.

Remark 2. Theorem 2 makes it possible to consider equations with coefficients depending on t, problems with fixed b, etc. (see [1]).

Example 1. $dx(t)/dt = A(t)x(t) + u(t), \ x(a) = x_0, \ x(b) = x_1, \ f^0 = 1,$ U is the ball of unit radius in the space B, strictly convex (i.e., if $\|u_1\| + \|u_2\| = 1$, then $\|u_1 + u_2\| < 2$, if $u_1 \neq u_2$), and $A(t)$ is a bounded linear operator in this space. From Theorem 2 follows the uniqueness of the optimal control.

Example 2. Let

$$\mathcal{L}x(t, s) = \sum_{i,j=1}^{n} a_{ij}(t, s) \frac{\partial^2 x(t, s)}{\partial s_i \partial s_j} + \sum_{i=1}^{n} b_i(t, s) \frac{\partial x(t, s)}{\partial s_i} + c(t, s)x(t, s) \qquad (3)$$

be an elliptic operator with sufficiently smooth coefficients, $s = (s_1, \dots, s_n) \in \Omega$, where Ω is a bounded domain with smooth boundary Γ in n-dimensional space, $a \leq t \leq b$.

Given the equation $\partial x(t, s)/\partial t = \mathcal{L}x(t, s) + f(t, s) + u(t, s)$ and the conditions: $x(a, s) = x_0(s) \in W_2^{(2)}(\Omega), \ x(b, s) = x_1(s) \in W_2^{(2)}(\Omega), \ x(t, s)|_\Gamma = 0$ (see [4]),

where

$$U = \left\{ u(t,s) : \int_\Omega u^2(t,s)\,ds \le 1 \right\}.$$

It is required to minimize the time $b - a$ of transition from $x_0(s)$ to $x_1(s)$, i.e. $f^0(x,u) \equiv 1$. It is not difficult to show the existence of an optimal control (provided there is at least one $u(t,s)$ for which there exists a solution with the given boundary conditions), and its uniqueness follows from Theorem 2 (see (2)).

3. Finally, consider the optimal-control problem when $x(b)$ belongs to a given convex body in a Banach space, for example,

$$\|x(b) - x_1\|_B \le \rho.$$

The time b is not fixed.

k_1) Let $A = R^1 \times B$. For almost all t, the solution of problem (1) satisfies $x(t) \in B$ and $f(x(t), u(t)) \in B$.

k_2) Let $\tilde{u}(t) \ne u(t)$ only when $\tau - \varepsilon < t < \tau$, and let $\tilde{u}(t) = v$ for these t (v is an arbitrary element of U), and let $\tilde{x}(t)$ be the solution of (1) corresponding to $\tilde{u}(t)$. On the interval (a,b) one can specify a set M of full measure of points τ for which

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[\tilde{y}(t) - y(t)] = z(t) \in A.$$

k_3) Let $a^* \in A^*$ be an arbitrary continuous linear functional on A. There exists a function $\psi(t) \in A^*$ ($a \le t \le b$) such that $\psi(b) = a^*$, $(\psi(t), z(t)) = \text{const}$ ($\tau \le t \le b$), and $\psi_0(t) = \text{const}$.

k_4) For every $\tau \in M$ there exist

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left[\tilde{y}_i(t + \varepsilon\theta_i(t,\tau)) - y_i(t)\right] = z_i(t) \in A \quad (i = 2,3),$$

where $\tilde{y}_i(t)$ and $\theta_i(t,\tau)$ are defined in i_1. Condition k_3) is fulfilled if $z(t)$ is replaced by $z_2(t)$ or $z_3(t)$.

Theorem 3. *If conditions $k_1), k_2), k_3)$ are fulfilled, then there exists $a^* \in A^*$ ($a^* \ne 0$) such that the function of u*

$$H(\psi(t), x(t), u) = (\psi(t), \varphi(x(t), u))$$

attains its maximum at $u = u(t)$, where $u(t)$ is the optimal control and $x(t)$ is the corresponding solution of problem (1). In this case $\psi_0 \le 0$. If condition k_4) is also fulfilled, then $H(\psi(t), x(t), u(t)) \equiv 0$.

Examples. If the function $f^0(x,u) = 1$, then Theorem 3 implies the uniqueness of the optimal control for the following problems:

$$1)\quad \frac{\partial x(t,s)}{\partial t} = \sum_{k=1}^{n} A_k(t,s)\frac{\partial x(t,s)}{\partial s_k} + B(t,s)x(t,s) + f(t,s) + u(t,s),$$

where $x = (x_1, \ldots, x_N)$, and the system is hyperbolic with smooth coefficients. The boundary conditions are prescribed in the form

$$x(a,s) = x_0(s) \in W_2^{(1)}(R^n), \qquad \|x(b,s) - x_1(s)\|_{\mathcal{L}_2(R^n)} \le \rho,$$

and the domain U is the ball:

$$\{u(s) : \|u(s)\|_{W_2^1(R^n)} \le 1\}.$$

The existence of an optimal control can be proved by using the well-known inequality of I. G. Petrovskii (3):

$$\|x(t,s)\|_{\mathcal{L}_2(R^n)} \le C\left\{\|x_0(s)\|_{\mathcal{L}_2(R^n)} + \|f(t,s) + u(t,s)\|_{\mathcal{L}_2(R^n\times(a,b))}\right\}.$$

2)

$$\frac{\partial^2 x(t,s)}{\partial t^2} = \mathcal{L}x(t,s) + f(t,s) + u(t,s),$$

where $\mathcal{L}x(t,s)$ is defined in (3). A mixed problem is considered in the cylinder $Q = \Omega \times [a,b]$ with the conditions

$$x(a,s) = x_0(s) \in W_2^{(2)}(\Omega), \qquad \frac{\partial x(a,s)}{\partial t} = x_1(s) \in W_2^{(1)}(\Omega), \qquad x(t,s)\big|_{\Gamma} = 0.$$

Moreover, for $t = b$ the inequality

$$\|x(b,s) - x_2(s)\|^2_{W_2^{(1)}(\Omega)} + \left\|\frac{\partial x(b,s)}{\partial t} - x_3(s)\right\|^2_{\mathcal{L}2(\Omega)} \le \rho^2,$$

is satisfied, and the domain

$$U = \{u(s) : \|u(s)\|_{W_2^{(1)}(\Omega)} \le 1\}.$$

3)

$$\frac{\partial x(t,s)}{\partial t} = \mathcal{L}x(t,s) + f(t,s) + u(t,s),$$

where $\mathcal{L}x(t, s)$ is defined in (3). The conditions on the boundary of Q have the form:

$$x(a, s) = x_0(s) \in W_2^{(2)}(\Omega), \qquad x(t, s)\big|_\Gamma = 0, \qquad \|x(b, s) - x_1(s)\|_{\mathcal{L}2(\Omega)} \le \rho,$$

and the domain

$$U = \{u(s) : \|u(s)\|_{W_2^{(1)}(\Omega)} \le 1\}.$$

The existence of optimal controls in examples 2) and 3) is easy to prove by using the known a priori estimates.

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Note: Figure translations are in progress. See original paper for figures.