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Abstract

Full Text

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ANALYTIC REPRESENTATION OF LINEAR FUNCTIONALS IN SPACES OF HARMONIC AND ANALYTIC FUNCTIONS CONTINUOUS IN A CLOSED DOMAIN

(Presented by Academician V. I. Smirnov on 12 II 1963)

Let γ be a plane curve homeomorphic to a circle; $\text{Int } \gamma$, $\text{Ext } \gamma$ respectively the interior and exterior of γ ; F a compact set in the complex plane; $C(F)$ the space of all complex functions continuous on F , with the usual norm; $\mathfrak{B}(F)$ the set of all Borel subsets of F ; $M(F)$ ($M^+(F)$, $M_r(F)$) the set of all (respectively nonnegative, real) finite complex countably additive measures defined on $\mathfrak{B}(F)$. Let G be a domain, $C_A(G)$ ($C_H(G)$) the subspace of $C(\overline{G})$ consisting of all functions continuous in \overline{G} and regular (harmonic) in G .

Every functional Φ , additive and continuous in $C_A(\text{Int } \gamma)$ ($C_H(\text{Int } \gamma)$), can be represented in the form

$$\Phi(x) = \int_{\overline{\text{Int } \gamma}} x d\mu \quad (x \in C_A, \text{ respectively } C_H), \quad (1)$$

where $\mu \in M(\overline{\text{Int } \gamma})$.

In this article we study the connection between the various measures μ that may occur in the representation (1) of a given functional Φ , and we characterize the functionals Φ that admit a representation (1) with a measure μ absolutely continuous with respect to harmonic measure. The results obtained in this direction are applied to the problem of representability of a regular function by an integral of Cauchy type (cf. (3)) and to the problem of simultaneous approximation of two different functions in different metrics by one and the same polynomial.

Let $z \in \text{Int } \gamma$, $e \in \mathfrak{B}(\gamma)$. By the symbol $\omega_z(e)$ we denote the harmonic measure of e with respect to $\text{Int } \gamma$, evaluated at the point z (1).

Theorem 1. Let $\mu \in M_r(\overline{\text{Int } \gamma})$. Then, for every $x \in C_H(\text{Int } \gamma)$,

$$\int_{\overline{\text{Int } \gamma}} x d\mu = \int_{\gamma} x d\nu, \quad (2)$$

where $\nu \in M_r(\gamma)$,

$$\nu(e) = \mu(e) + \int_{\text{Int } \gamma} \omega_z(e) d\mu_z \quad (e \in \mathfrak{B}(\gamma)), \quad (3)$$

$$|\nu|(\gamma) = |\mu|(\overline{\text{Int } \gamma})$$

($|\nu|$ and $|\mu|$ denote, respectively, the total variations of the measures ν and μ).

If $\nu_1 \in M_r(\gamma)$ and

$$\int_{\text{Int } \gamma} x d\mu = \int_{\gamma} x d\nu_1$$

at least for all $x \in C_A(\text{Int } \gamma)$, then $\nu_1 = \nu$.

The first assertion is verified by direct computation; the second follows from the fact that the measure $\nu - \nu_1$ is orthogonal to all harmonic polynomials, which are everywhere dense in $C(\gamma)$ (2).

If the curve γ is rectifiable, then by $L_p(\gamma)$ ($p \geq 1$) we denote the space of all complex functions x defined on γ and such that

$$\|x\|_{L_p(\gamma)}^p = \int_{\gamma} |x(\zeta)|^p |d\zeta| < +\infty.$$

Let G be a domain; $L_p(G)$ is the space (with the usual norm) of all complex functions summable in G to the p -th power with respect to planar Lebesgue measure. The symbols $L_\infty(\gamma)$, $L_\infty(G)$ are also to be understood in the usual way. The symbols $L_p^r(G)$, $L_p^r(\gamma)$ denote the set of all real-valued functions from $L_p(G)$, $L_p(\gamma)$, respectively, $1 \leq p \leq +\infty$.

Theorem 2. Let γ be rectifiable.

1. If $\Psi \in L_1(\text{Int } \gamma)$, $\Psi \geq 0$, then there exists a unique function $\psi \in L_1(\gamma)$, $\psi \geq 0$, such that

$$\int_{\text{Int } \gamma} \varphi \Psi dx dy = \int_{\gamma} \varphi \psi |d\zeta| \quad (4)$$

for all $\varphi \in C_H(\text{Int } \gamma)$; moreover,

$$\|\Psi\|_{L_1(\text{Int } \gamma)} = \|\psi\|_{L_1(\gamma)}.$$

2. If $\psi \in L_1(\gamma)$ is real-valued, and $\varepsilon > 0$, then there exists a real-valued function $\Psi \in L_1(\text{Int } \gamma)$ such that

$$\|\Psi\|_{L_1(\text{Int } \gamma)} < \|\psi\|_{L_1(\gamma)} + \varepsilon,$$

and (4) is satisfied.

The **proof** of assertion 1 follows immediately from Theorem 1, where one must put

$$\mu(e) = \int_e \Psi \, dx \, dy.$$

The correspondence $\mu \rightarrow \nu$ established in Theorem 1 defines a linear operator T from $L_1^r(\text{Int } \gamma)$ into $L_1^r(\gamma)$. The operator adjoint to this operator turns out to be an operator A , which assigns to a function $f \in L_\infty^r(\gamma)$ the harmonic function $F \in L_\infty^r(\text{Int } \gamma)$ having angular boundary values almost everywhere on γ equal to f . The operator A maps $L_\infty^r(\gamma)$ isometrically onto a closed subspace of $L_\infty^r(\text{Int } \gamma)$. Hence the operator T realizes an isometric mapping of

$$L_1^r(\text{Int } \gamma) / [AL_\infty^r(\gamma)]^\perp$$

onto all of $L_1^r(\gamma)$, which is what had to be proved. Here $[AL_\infty^r(\gamma)]^\perp$ is the set of all such $\Psi \in L_1^r(\text{Int } \gamma)$ that

$$\int_{\text{Int } \gamma} \Psi \varphi \, dx \, dy = 0$$

for all harmonic $\varphi \in L_\infty(\text{Int } \gamma)$.

Let us also note that for every $\mu \in M^+(\text{Int } \gamma)$ such that $\mu(\gamma) = 0$, it is easy to construct a $\Psi \in L_1(\text{Int } \gamma)$ such that $\Psi \geq 0$,

$$\|\Psi\|_{L_1(\text{Int } \gamma)} = \mu(\text{Int } \gamma),$$

$$\int_{\text{Int } \gamma} \varphi \, d\mu = \int_{\text{Int } \gamma} \varphi \Psi \, dx \, dy$$

for all $\varphi \in C_H(\text{Int } \gamma)$.

Theorem 3. Let Φ be an additive functional defined on $C_A(\text{Int } \gamma)$, $e \subset \gamma$, $e \in \mathfrak{B}(\gamma)$, $\omega_z(e) = 0$. In order that Φ admit a representation

$$\Phi(x) = \int_\gamma x \, d\nu \quad (x \in C_A(\text{Int } \gamma)), \quad (5)$$

where $\nu \in M(\gamma)$, $|\nu|(e) = 0$, it is necessary and sufficient that the functional Φ be continuous in the following sense: $\Phi(x_n) \rightarrow 0$ whenever the sequence of functions x_n , regular in $\text{Int } \gamma$, is such that $x_n(\zeta) \rightarrow 0$ for all $\zeta \in \gamma \setminus e$, and

$$\sup_n \|x_n\|_{C(\gamma)} < +\infty.$$

In the necessity part the theorem is obvious; in the sufficiency part its proof is easy to obtain by using the extendability of any function from $C(e)$ to a function from $C_A(\text{Int } \gamma)$ (4, 5).

Let Φ be a linear functional on $C(\overline{\text{Int } \gamma})$ that vanishes on $C_A(\text{Int } \gamma)$. What measures μ can correspond to such a functional?

An exhaustive answer to this difficult and important question has been given only in the case when the measure μ is concentrated on γ (6). Therefore the following theorem may be of some interest.

Theorem 4. Let $\mu \in M(\overline{\text{Int } \gamma})$,

$$\int_{\overline{\text{Int } \gamma}} x d\mu = 0 \quad \text{for all } x \in C_A(\overline{\text{Int } \gamma}).$$

1. If $\text{Re } \mu(e) \geq 0$ for all $e \in \mathfrak{B}(\gamma)$, then $\text{Re } \mu = 0$.
2. If all values of μ are contained inside or on the sides of some angle of aperture smaller than π , with vertex at the origin, then $\mu = 0$.
3. If $e \in \mathfrak{B}(\gamma)$, $\omega_z(e) = 0$, then $\mu(e) = 0$.
4. Let $\mathcal{A} \subset M(\overline{\text{Int } \gamma})$, and let every function $x \in C(\overline{\text{Int } \gamma})$ satisfying the condition

$$\int_{\overline{\text{Int } \gamma}} x d\nu = 0 \quad \text{for all } \nu \in \mathcal{A}$$

belong to $C_A(\overline{\text{Int } \gamma})$. Then μ belongs to the closure of the linear hull of \mathcal{A} in the weak $*$ -topology of the space $M(\overline{\text{Int } \gamma})$.

As the set \mathcal{A} , by Morera's theorem, one may take the totality of all measures ν_Γ of the form

$$\nu_\Gamma(e) = \int_{\Gamma \cap e} dx + i dy \quad (e \in \mathfrak{B}(\overline{\text{Int } \gamma})),$$

where Γ runs through the set of all triangular oriented contours lying in $\overline{\text{Int } \gamma}$.

The simplest examples show that a measure μ satisfying the condition of the theorem may be real and nonzero on one-point sets (of course, only when

$$|\mu|(\gamma) < |\mu|(\overline{\text{Int } \gamma}).$$

)

From Theorems 2 and 3 it follows easily that

Theorem 5. Let the function u be regular in $\text{Ext } \gamma$, $u(\infty) = 0$, and let γ be rectifiable. The following assertions are equivalent:

1)

$$u(z) = \int_{\text{Int } \gamma} \frac{d\mu}{\xi - z}, \quad \text{where } \mu \in M(\overline{\text{Int } \gamma}), \quad |\mu|(\gamma) = 0.$$

2)

$$u(z) = \int_{\text{Int } \gamma} \frac{\Psi dx dy}{\xi - z}, \quad \Psi \in L_1(\text{Int } \gamma).$$

3)

$$u(z) = \int_{\gamma} \frac{\psi(\xi) |d\xi|}{\xi - z}, \quad \psi \in L_1(\gamma).$$

4) The functional Φ_u , defined on the set of all f regular on $\overline{\text{Int } \gamma}$ by the formula

$$\Phi_u(f) = \int_{\gamma_f} u(\xi) f(\xi) d\xi,$$

satisfies the following condition: if the functions x_n are regular in $\overline{\text{Int } \gamma}$,

$$\lim_{n \rightarrow \infty} x_n(\zeta) = 0$$

for almost all $\zeta \in \gamma$,

$$\sup_n \|x_n\|_{C(\gamma)} < \infty,$$

then

$$\lim_{n \rightarrow \infty} \Phi_u(x_n) = 0.$$

Here γ_f is such a rectifiable Jordan curve that

$$\text{Int } \gamma_f \supset \overline{\text{Int } \gamma}, \quad f \in C_A(\overline{\text{Int } \gamma_f}).$$

Corollary. Let γ_1 and γ be rectifiable Jordan curves,

$$\gamma_1 \subset \text{Int } \gamma, \quad \psi_1 \in L_1(\gamma_1).$$

There exists $\psi \in L_1(\gamma)$ such that

$$\int_{\gamma_1} \frac{\psi_1 d\xi}{\xi - z} = \int_{\gamma} \frac{\psi d\xi}{\xi - z} \quad \text{for all } z \in \text{Ext } \gamma \quad (\text{cf. (3)}).$$

This corollary may be useful in studying boundary properties of Cauchy-type integrals. We note that the assertions obtained from assertions 3) and 2) of Theorem 5 by replacing L_1 with L_p ($p > 1$) are not equivalent.

With the aid of Theorem 3 we shall prove the following approximation theorem.

Theorem 6. Let γ and e denote the same as in Theorem 3. Let X be a normed space consisting of functions regular in $\text{Int } \gamma$ and containing $C_A(\text{Int } \gamma)$. Let

$$\lim_{n \rightarrow \infty} \|x_n\|_X = 0$$

for any sequence of the sequence $x_n \in X$, satisfying the conditions of Theorem 3. Let $f \in \overline{P}$, where P is the set of all polynomials, and the closure is taken in the topology of X . Let $\varphi \in C(e)$, $\varepsilon > 0$. Then there exists a $p \in P$ such that $\|f - p\|_X + \|\varphi - p\|_{C(e)} < \varepsilon$.

For the proof, note that a linear functional Φ , continuous in $X \times C(e)$, has the form

$$\Phi((x, y)) = \int_{\gamma} x d\mu + \int_{\gamma} y d\nu,$$

where $\mu, \nu \in M(\gamma)$, $|\mu|(e) = 0$, $|\nu|(\gamma \setminus e) = 0$, $x \in C_A(\text{Int } \gamma)$, $y \in C(e)$. If $\Phi((p, p)) = 0$ for all $p \in P$, then $|\mu + \nu|(e) = 0$, and hence $\nu = 0$, $\Phi((f, \varphi)) = 0$.

We give examples of spaces X to which Theorem 6 is applicable.

1. X is the space of all x , regular in $\text{Int } \gamma$, such that

$$\int_{\text{Int } \gamma} |x|^p dH = \|x\|_X^p < +\infty,$$

where $p \geq 1$, $H \in M^+(\text{Int } \gamma)$, $H(e) = 0$. For $p = 2$ and $dH = dx dy$ we obtain a theorem of E. M. Kegeyan ⁽⁷⁾.

2. X is the space of all x , regular in $\text{Int } \gamma$, such that

$$\|x\|_X = \sup_{p \geq 1} \lambda(p) \|x\|_{L_p(\text{Int } \gamma)} < +\infty,$$

where the function λ is continuous, positive on $[1, +\infty)$, and $\lim_{p \rightarrow +\infty} \lambda(p) = 0$.

3. Let γ be the unit circle. Then for X one may take the space of all functions x , regular in $\text{Int } \gamma$, such that

$$\|x\|_X^p = \sum_{n=0}^{\infty} \frac{|x^{(n)}(0)|^p}{(n!)^p} < +\infty, \quad p \geq 2.$$

4. Let γ be the unit circle; X consists of all x , regular in $\text{Int } \gamma$, such that

$$\|x\|_X = \sup_{|z| < 1} \Lambda(|z|) |x(z)| < +\infty,$$

where Λ is a positive continuous function defined on $[0, 1)$, $\lim_{\rho \rightarrow 1} \Lambda(\rho) = 0$. This space was considered in ⁽⁸⁾.

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* An analogous result was obtained by E. Yu. Kegeyan in a more general case (for any Carathéodory domain).

Note: Figure translations are in progress. See original paper for figures.

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