A VARIATIONAL PRINCIPLE IN THE GENERAL THEORY OF RELATIVITY

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Abstract

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A VARIATIONAL PRINCIPLE IN THE GENERAL THEORY OF RELATIVITY

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Usually the equations of the gravitational field and the conservation laws of energy–momentum are found by varying the so-called pseudoscalar, containing only the first derivatives g_{ik}, and assuming that on some hypersurface S_l the variations $\delta\Gamma^l_{ik} = 0$ ([1]).

The energy carried by gravitational waves is different in different reference systems and is equal to zero in an inertial system, which has led some researchers even to doubt the real existence of gravitational waves ([2]). In other variants of the general theory of relativity ([3–5]), other expressions for t_i^k are sought, sometimes symmetric in the indices i, k, sometimes with second derivatives G_{ik}, but these variants have proved to be very artificial and unsatisfactory.

We shall seek the equations of the gravitational field and the conservation laws of energy–momentum by varying the density of the scalar curvature:

$$R = R\left(g_{ik};\ \frac{\partial g_{ik}}{\partial x^l};\ \frac{\partial^2 g_{ik}}{\partial x^l \partial x^r}\right). \tag{1}$$

In doing so we shall not discard variations with respect to the second derivatives of g_{ik}, and shall not assume that on the hypersurface S_l the $\delta\Gamma^l_{ik}$ are equal to zero. Already in classical theory it is proved that on any surface bounding the region of a given gravitating medium, the variations of the acceleration are not equal to zero (if the density there is not equal to zero); in the general theory of relativity the quantities Γ^i_{ik} play the role of accelerations, and g_{ik} the role of potentials. Consequently, we have every reason, when considering a field with sources of the field–matter, to assume that on S_l, $\delta\Gamma^l_{ik} \neq 0$.

To obtain the field equation we shall use the variational equation generalized to second derivatives, varying $L = -R/2\varkappa$ (in deriving this equation we also assume that on the boundaries of integration $\delta\Gamma^l_{ik} \neq 0$, and take into account that sources of the field–matter–are present in the field). Then we arrive at the equation

$$\delta g^{ik}\left[\frac{\partial(\sqrt{-g}L)}{\partial g^{ik}}-\frac{\partial}{\partial x^l}\frac{\partial(\sqrt{-g}L)}{\partial g^{ik}/\partial x^l}+\frac{\partial^2}{\partial x^l\partial x^r}\frac{\partial(\sqrt{-g}L)}{\partial^2 g^{ik}/\partial x^l\partial x^r}+\frac{\sqrt{-g}}{2}T_{ik}\right]=0, \tag{2}$$

where

$$\frac{\sqrt{-g}}{2}T_{ik}\delta g^{ik}=\frac{\partial}{\partial x^l}\left[\left(\frac{\partial(\sqrt{-g}L)}{\partial g^{ik}/\partial x^l}-\frac{\partial}{\partial x^r}\frac{\partial(\sqrt{-g}L)}{\partial^2 g^{ik}/\partial x^l\partial x^r}\right)\delta g^{ik}+\frac{\partial(\sqrt{-g}L)}{\partial^2 g^{ik}/\partial x^l\partial x^r}\delta\frac{\partial g^{ik}}{\partial x^r}\right]. \tag{3}$$

the components of the tensor T_{ik} determine the sources of the field—the matter. Here

$$\sqrt{-g}\,L=\mathcal{L}=-\frac{\sqrt{-g}}{2\varkappa}\,R \tag{4}$$

is the density of the Lagrange function—the Lagrangian,

$$\varkappa=8\pi\mathcal{G}/c^4 \tag{5}$$

is Einstein' s gravitational constant, $\mathcal{G}$ is the usual gravitational constant.

Let us now represent the variation $\delta(\sqrt{-g}\,R)$ in the form

$$\begin{aligned}\delta(\sqrt{-g}\,R)&=\delta g^{ik}\left[\sqrt{-g}\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\right]+\delta\frac{\partial^2 g^{ik}}{\partial x^l\partial x^m}\left[\sqrt{-g}\,(g^{lm}g_{ik}-\delta^l_i\delta^m_k)\right]\\&\quad+\frac{1}{2}\,\delta g^{ik}\left\{\frac{\partial}{\partial x^l}\left[\sqrt{-g}\left(3g^{ml}\frac{\partial g_{ik}}{\partial x^m}-2g^{ml}\frac{\partial g_{im}}{\partial x^k}-g^{rm}\frac{\partial g_{rm}}{\partial x^i}\delta^l_k\right)\right]\right\}\\&\quad+\frac{1}{2}\,\delta\frac{\partial g^{ik}}{\partial x^l}\left[\sqrt{-g}\left(3g^{ml}\frac{\partial g_{ik}}{\partial x^m}-2g^{ml}\frac{\partial g_{im}}{\partial x^k}-2g^{rm}\frac{\partial g_{rm}}{\partial x^i}\delta^l_k\right)+2\frac{\partial}{\partial x^m}\left[\sqrt{-g}\,g^{lm}g_{ik}\right]\right]\\&=\delta g^{ik}\left[\sqrt{-g}\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\right]+\frac{\partial}{\partial x^l}\left[\sqrt{-g}\,(g^{ik}\delta\Gamma^l_{ik}-g^{il}\delta\Gamma^k_{ik})\right].\end{aligned} \tag{6}$$

Using this expression, we compute from formula (2) the values of the derivatives and find that

$$R_{ik}-\frac{1}{2}g_{ik}R=\varkappa T_{ik}, \tag{7}$$

where now T_{ik} can indeed be identified with the tensor of matter.

Substitution of the second, third, and fourth terms of equation (6) into (2) gives a zero result (i.e., the terms obtained from $\sqrt{g}\, g^{ik}\delta R_{ik}$ give no contribution upon variation).

Next, it is easy to show, using (4), that

$$-\varkappa\sqrt{-g}\,T_{ik}\delta g^{ik} = \sqrt{-g}\,g^{ik}\delta R_{ik}. \tag{8}$$

(Hence it is obvious that T_{ik} is indeed a tensor.)

Let us write this equation in the form

$$\sqrt{-g}\,(g^{ik}\delta R_{ik} + \varkappa T_{ik}\delta g^{ik}) = -2\varkappa\,\delta(\sqrt{-g}\,\hat{p}) = 0, \tag{9}$$

where $\hat{p}$ may be identified with the total pressure. Putting in equation (2)

$$L_m = \hat{p}, \tag{10}$$

we arrive at the identity $\frac{\sqrt{-g}}{2}T_{ik} \equiv \frac{\sqrt{-g}}{2}T_{ik}$, which verifies the validity of equation (9). The plus sign in relation (10) is natural, since the pressure of the medium is opposite in sign to attraction, which is characterized by the Lagrangian $L = -R/2\varkappa$.

Let us note that the part of the energy-momentum tensor

$$T_{mik} = (p+\varepsilon)u_i u_k + g_{ik}p,$$

characterizing matter consisting of particles with nonzero rest mass, can be obtained from the Lagrangian

$$\begin{aligned} L = p &= \frac{1}{4}\left[(p+\varepsilon)u_i u_k u_l u_m g^{il}g^{km} + (3p-\varepsilon)\right] \\ &= \frac{1}{v}\left[-E + c\sqrt{-g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k}}\right], \end{aligned} \tag{11}$$

where $c\,\partial\bar{S}/\partial x^i = wu_i$ ($\bar{S}$ is the action), $w = (p+\varepsilon)v$ is the heat content, substituting this Lagrangian into equation (2). In an analogous way one can also obtain the energy-momentum tensor of the electromagnetic field

$$T_{ik} = \frac{1}{4\pi}\left[F_{il}F_k^l - \frac{1}{4}g_{ik}F_{lm}F^{lm}\right]$$

from the Lagrangian

$$L = -\frac{1}{16\pi} F_{ik} F_{lm} g^{il} g^{km}. \tag{12}$$

Let us now compute the quantity

$$\frac{\partial(\sqrt{-g}L)}{\partial x^i} = \frac{\partial(\sqrt{-g}L)}{\partial g^{ml}} \frac{\partial g^{ml}}{\partial x^i} + \frac{\partial(\sqrt{-g}L)}{\partial\, \partial g^{ml}/\partial x^k} \frac{\partial^2 g^{ml}}{\partial x^i \partial x^k} + \frac{\partial(\sqrt{-g}L)}{\partial\, \partial^2 g^{ml}/\partial x^k \partial x^r} \frac{\partial^3 g^{ml}}{\partial x^i \partial x^k \partial x^r}. \tag{13}$$

Eliminating from (2) and (13) the quantities $\partial(\sqrt{-g}L)/\partial g^{ml}$, we arrive at the equation

$$\frac{\partial(\sqrt{-g}\, t_i^k)}{\partial x^k} = \frac{\sqrt{-g}}{2} T_{lm} \frac{\partial g^{lm}}{\partial x^i} = -\frac{\sqrt{-g}}{2} T^{lm} \frac{\partial g_{lm}}{\partial x^i}. \tag{14}$$

Further,

$$\frac{\partial}{\partial x^k} \left[\sqrt{-g}\, (T_i^k + t_i^k)\right] = \frac{\partial}{\partial x^k} \left[\sqrt{-g} \left(\bar{t}_i^k + \frac{R_i^k}{\varkappa}\right)\right] = 0. \tag{15}$$

Calculations give the following value of the components $\bar{t}_i^k$:

$$-2\varkappa \bar{t}_i^k = \left(\frac{\partial^2 g^{lm}}{\partial x^i \partial x^r} g_{lm} g^{rk} - \frac{\partial^2 g^{kr}}{\partial x^i \partial x^r}\right) +$$

$$+\frac{1}{2}\left[\frac{\partial g^{lm}}{\partial x^i} g^{rk} \left(3\frac{\partial g_{lm}}{\partial x^r} - 2\frac{\partial g_{lr}}{\partial x^m}\right) - g^{lm} \frac{\partial g_{lm}}{\partial x^r} \frac{\partial g^{rk}}{\partial x^i}\right] = g^{lm} \frac{\partial \Gamma^k_{ml}}{\partial x^i} - g^{km} \frac{\partial \Gamma^l_{ml}}{\partial x^i} \tag{16}$$

or

$$-2\varkappa \bar{t}_{ik} = g_{kr} g^{lm} \frac{\partial \Gamma^r_{ml}}{\partial x^i} - \frac{\partial \Gamma^l_{kl}}{\partial x^i}. \tag{17}$$

These equations are a consequence of the Bianchi identities, since, using them, one can bring (14) to the form (15).

The pseudotensors t_i^k or $\bar{t}_i^k$ are not symmetric in the indices $i; k$; therefore the conservation laws for angular momentum and energy are not fulfilled; the nonzero divergence in equation (6) is interpreted as the source of the field–matter.

If, when varying $L = -R/2\varkappa$ on S_l, one sets $\delta\Gamma^l_{ik} = 0$, then it is also necessary to vary the matter Lagrangian $L_m = p$; in this case all our results retain their force. However, in order to avoid an unnecessary and unjustified dualism, it proved more correct and proper to consider the variational problem and seek the equation of the gravitational field in the presence of sources of this field, varying only one Lagrangian and not imposing on the hypersurface S_l that $\delta\Gamma^l_{ik} \neq 0$.

This requirement is equivalent to introducing into the Lagrange equation an additional term characterizing matter and "correcting" the extremal, which is what we have done. The derivation of the field equation from a unified Lagrangian is natural, since the gravitational field and matter are unified. As a result of this unified generalized variational formalism we have also obtained the correct energy-momentum pseudotensor of the field itself and exact conservation laws (identities) for the energy-momentum of matter and the field. (This

pseudotensor differs from the Møller-Mickevich pseudotensor by the derivative of the antisymmetric pseudotensor χ_i^{kl}, which in our case is half as large as theirs.) We have

$$\chi_i^{kl} = \sqrt{-g}\, g^{kn} g^{lm} \left(\frac{\partial g_{im}}{\partial x^n} - \frac{\partial g_{in}}{\partial x^m} \right);$$

$$\sqrt{-g}\,(T_i^k + t_i^k) = \frac{1}{2\varkappa} \frac{\partial \chi^l_{ik}}{\partial x^l}.$$

As we have already indicated, these laws coincide with the Bianchi identities, which corresponds to the absence of nontrivial groups of transformations of the field equations (7), i.e., to a completely inhomogeneous space. In this case the gravitational interactions are minimally weak interactions, and conservation laws of momenta cannot hold. The symmetrization of t_i^k can formally be carried out, but this is not a very useful operation. (We note that in a rigorous variational method of deriving the field equations, the so-called λ-term is absent from these equations.)

The pseudotensor t_i^k derived by us is the only one for which the Bianchi identities are satisfied and real gravitational waves exist, carrying energy in all reference systems and, in particular, in inertial ones, and for which the energy is defined uniquely. The exception is provided by definitely chosen noninertial reference systems, in which proper waves are emitted.

In conclusion, let us note that in the case of plane gravitational waves the quantity t_{01}, characterizing the energy flux of gravitational waves, will be determined by the simple relation:

$$-2\varkappa t_{01} = -2\varkappa \bar{t}_{01} = \frac{1}{2} \frac{\partial g^{ml}}{\partial x^1} \frac{\partial g_{lm}}{\partial x^0} - \frac{\partial^2 \ln g}{\partial x^1 \partial x^0} = 2R_{01}. \tag{18}$$

In the case of weak waves we shall have

$$t_{01} = -t^{01} = \frac{c^3}{32\pi\mathcal{G}}\left[\left(\frac{\partial h_{22}}{\partial t}\frac{\partial h_{22}}{\partial x} + \frac{\partial h_{33}}{\partial t}\frac{\partial h_{33}}{\partial x} + 2\frac{\partial h_{23}}{\partial t}\frac{\partial h_{23}}{\partial x}\right) - \frac{\partial^2}{\partial t\,\partial x}\left(h_{22}^2 + h_{33}^2 + 2h_{23}^2\right)\right], \tag{19}$$

where $h_{22} = g_{22} - 1$, $h_{33} = g_{33} - 1$, $h_{23} = g_{23}$, with $h_{22} = -h_{33}$, and the quantities h_{22} and h_{23} are of the same order.

Since, in the quadrupole radiation of gravitational waves,

$$h_{22}^2 + h_{33}^2 + 2h_{23}^2 = 2\left(h_{22}^2 + h_{33}^2\right) = \text{const},$$

then

$$t_{01} = -t^{01} = \frac{c^3}{16\pi\mathcal{G}}\left[\frac{\partial h_{22}}{\partial t}\frac{\partial h_{22}}{\partial x} + \frac{\partial h_{23}}{\partial t}\frac{\partial h_{23}}{\partial x}\right], \tag{20}$$

which agrees with Einstein’ s classical results.

In conclusion, let us note that, apparently, the idea of interpreting a zero divergence of the form (3) as a source of the field was first advanced by A. Eddington ([6]). However, for this purpose he used only the Schwarzschild field.

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CITED LITERATURE

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[5] N. W. Mitzkewitch, *Ann. der Phys.*, **1**, 319 (1958).
[6] A. Eddington, *The Mathematical Theory of Relativity*, 1934, § 63.

Note: Figure translations are in progress. See original paper for figures.