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Abstract

Full Text

MATHEMATICS

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ENERGY INEQUALITIES FOR ELLIPTIC OPERATORS WITH DISCONTINUOUS COEFFICIENTS AND GENERAL BOUNDARY CONDITIONS AND CONJUGATION CONDITIONS

(Presented by Academician S. L. Sobolev on 20 VII 1962)

1°. It is known that energy inequalities of the form $\|u\|_l^2 \leq K(\|Au\|_0^2 + \|u\|_0^2)$ play an important role in proofs of existence, uniqueness, and smoothness of solutions of partial differential equations. Recently, in papers (1-7), energy inequalities with a boundary norm were proved for a number of boundary-value problems; their use gives a number of advantages.

In the present note the authors prove energy inequalities with a boundary norm for elliptic operators with discontinuous coefficients and general boundary conditions and conjugation conditions on surfaces of discontinuity. The method of proof is a development of the method applied by M. Schechter in (7). The inequalities obtained make it possible to investigate the solvability of general boundary-value problems for elliptic equations with discontinuous coefficients.

2°. Let G be a bounded domain of the n -dimensional Euclidean space E_n with boundary Γ ; let G_1 be a subdomain of G with boundary γ , having no common points with Γ ; $G_2 = G/\overline{G_1}^*$. Consider the direct sum of Sobolev spaces $W_2^l(G_1) + W_2^l(G_2) = W_2^l(G) = W_2^l$; obviously, W_2^l is obtained by taking the closure in the norm

$$\|u\|_1^2 = \int_G \sum_{|\alpha| \leq l} |D^\alpha u|^2 dx$$

of the set of functions defined in G and infinitely differentiable in each of the $\overline{G_i}$; in particular, $W_2^0(G) = L_2(G)$. Every function $u \in W_2^l$ can be represented in the form $u(x) = u_1(x) + u_2(x)$, where $u_i(x) = u(x)$, $x \in G_i$; $u_i(x) = 0$, $x \in G/\overline{G_i}$ ($i = 1, 2$). If $l > 0$, then for $x \in \gamma$ by $u_i(x)$ we mean the limiting value of $u(x)$ from the side of G_i .

Consider an elliptic differential operator A with discontinuous complex coefficients

$$(Au)(x) = \begin{cases} (A^1u)(x), & x \in G_1, \\ (A^2u)(x), & x \in G_2, \end{cases} \quad (1)$$

where

$$A^i = \sum_{|\mu| \leq 2m} a_\mu^i(x) D^\mu, \quad x \in G_i \quad (i = 1, 2); \quad (2)$$

$$\mu = (\mu_1, \dots, \mu_n), \quad |\mu| = \mu_1 + \dots + \mu_n; \quad D^\mu = D_1^{\mu_1} \dots D_n^{\mu_n}, \quad D_k = \frac{1}{i} \frac{\partial}{\partial x_k}.$$

The ellipticity condition means that, for every real vector $\xi = (\xi_1, \dots, \xi_n) \neq 0$, the characteristic polynomials

$$D^i(x, \xi) = \sum_{|\mu|=2m} a_\mu^i(x) \xi^\mu$$

are nonzero in G_i ($i = 1, 2$).

* The case of two domains is considered only to simplify the notation. All results are also valid for a partition into a finite number of domains.

Let x be any fixed point on $\Gamma(\gamma)$; consider the polynomial $P^i(\eta) = P^i(x, \tau + \eta\nu)$, where $\tau \neq 0$ is any real vector tangent to $\Gamma(\gamma)$ at the point x , and $\nu \neq 0$ is any real vector normal to $\Gamma(\gamma)$ at x . From ellipticity it follows that $P^i(\eta)$ have no real roots; we shall require that the complex roots of these polynomials be distributed equally in the upper and lower half-planes (in the case $n \geq 3$ this condition is always fulfilled⁽⁸⁾); moreover, $P^i(\eta) = P^i + (\eta)P^i - (\eta)$, where $P^i + (P^i -)$ are polynomials of degree m , all of whose roots lie in the upper (lower) half-plane.

We shall also introduce the boundary operators

$$B_k^i = \sum_{|\mu| \leq m_k^i} b_{k\mu}^i(x) D^\mu \quad (3)$$

$$(i = 1, 2, 3; \quad k = 1, \dots, r_i; \quad r_1 = r_2 = r \geq 2m; \quad r_3 \geq m;$$

$$m_k^1 = m_k^2 = m_k; \quad m_k^i \leq 2m - 1),$$

where the complex functions $b_{k\mu}^i(x)$ ($i = 1, 2$) are defined on γ , and $b_{k\mu}^3(x)$ on Γ ; denote their characteristic polynomials by

$$Q_k^i(x, \xi) = \sum_{|\mu|=m_k^i} b_{k\mu}^i(x) \xi^\mu;$$

if x is a fixed point on $\Gamma(\gamma)$, then, as above for $P^i(x, \xi)$, set $Q_k^i(\eta) = Q_k^i(x, \tau + \eta\nu)$.

Definition 1. Let at any point $x \in \gamma$, for any $\tau \neq 0$, $\nu \neq 0$, the following condition be satisfied: there exist natural numbers k_1, \dots, k_{2m} such that the simultaneous fulfillment of the identities

$$c_1 Q_{k_1}^i(\eta) + \dots + c_{2m} Q_{k_{2m}}^i(\eta) = C^i(\eta) P_{\pm}^i(\eta) \quad (i = 1, 2),$$

where c_1, \dots, c_{2m} are complex constants and $C^1(\eta), C^2(\eta)$ are polynomials, is possible only when $c_1 = \dots = c_{2m} = 0$; in other words,

$$\begin{pmatrix} Q_{k_t}^1 \\ Q_{k_t}^2 \end{pmatrix} \quad (t = 1, \dots, 2m)$$

are linearly independent modulo

$$\begin{pmatrix} P^1_+ \\ P^2_- \end{pmatrix}.$$

Then we shall say that the operators B_k^i ($i = 1, 2$) **jointly cover** the operators A^1, A^2 .**

This definition is analogous to the definition of covering systems of operators (6,7,9), according to which the boundary operators B_1, \dots, B_r ($r \geq m$) cover an elliptic operator A of order $2m$ if among the characteristic polynomials $Q_1(\eta), \dots, Q_r(\eta)$ there are m linearly independent modulo $P_+(\eta)$.

Definition 2. We shall say that the boundary operators (3) **cover** the operator A with discontinuous coefficients if the operators B_k^3 cover A^2 , and the operators B_k^1, B_k^2 jointly cover A^1, A^2 .

3°. Theorem. Let $a_\mu^i(x) \in C^s(\overline{G_i})$, $b_{k\mu}^i(x) \in C^{2m-m_k+s}(\gamma)$ ($i = 1, 2$); $b_{k\mu}^3(x) \in C^{2m-m_k^3+s}(\Gamma)$; let the surfaces Γ and γ be of class C^{2m+s} , where $s \geq 0$ is an integer. Then, and only then, when the operator A is elliptic and the boundary operators (3) cover it (in the sense of Definition 2), there exists a constant $K > 0$ such that

$$\|u\|_{2m+s}^2 \leq K \left(\|Au\|_s^2 + \sum_{k=1}^r \langle |B_{ku}| \rangle_{2m-m_k+s-1}^2 + \sum_{k=1}^{r_3} \langle B_{ku}^3 \rangle_{2m-m_k^3+s-1}^2 + \|u\|_0^{(2)} \right),$$

$$u \in W_2^{2m+s}(G). \quad (4)$$

* For $i = 1$ the plus subscript is taken here, and for $i = 2$ the minus subscript.

** It is easy to show that in this definition one may replace P^1+ , P^2- by P^1- , P^2+ .

Here $[B_k u](x) = (B_k^1 u_1)(x) - (B_k^2 u_2)(x)$, $x \in \gamma$; $\langle v \rangle_l$ is the boundary norm ⁽⁷⁾, defined for functions given on $\Gamma(\gamma)$ (in ⁽⁹⁾ this same norm is denoted by $\langle v \rangle_{l+1}$). Such norms were introduced earlier by L. N. Slobodetskii in ^(1, 2). One may assume, for example, that in the notation of ^(1, 2)

$$\langle v \rangle_l = \|v\|_{W_2^{l+1/2}(\Gamma)}$$

(respectively, for γ).

The proof of sufficiency is first carried out locally; in the general case the inequality is obtained by means of a partition of unity. Suppose, for example, that $x \in \gamma$; it is enough to prove that there exists a neighborhood U of the point x such that inequality (4) holds for all functions $u \in W_2^{2m+s}(U)$ that vanish outside U . In doing so it is sufficient to consider the case where U is an n -dimensional ball, $U \cap \gamma$ is a piece of the hyperplane $x_n = 0$, and the operators

$$A^i = \sum_{|\mu|=2m} a_{\mu}^i D^{\mu}, \quad B_k^i = \sum_{|\mu|=2m-1} b_{k\mu}^i D^{\mu}$$

have constant coefficients (cf. (7)). The proof of this for $s = 0$ is carried out with the aid of the Fourier transform; here the condition of joint covering is used essentially. This condition, by means of an algebraic apparatus that is a development of the algebraic methods used in ⁽⁷⁾, makes it possible to apply the lemma of Aronszajn ^(10, 11, 7). To obtain the inequality for $s = 1$, we write the obtained inequality for difference quotients in the tangential direction and, after passing to the limit, obtain estimates of the derivatives in this direction; estimates for the normal derivatives are obtained by differentiating Au and using the inequalities already found. The argument is analogous for $s = 2, 3, \dots$

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