MATHEMATICS

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1963

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Abstract

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THE WEIERSTRASS THEOREM IN SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

(Presented by Academician A. N. Kolmogorov, 14 I 1963)

1°. In a report at the Fourth All-Union Mathematical Congress (Leningrad, 1961), the author of the present note showed that certain special rings of infinitely differentiable functions possess a peculiar structure of the set of primary ideals, forming transfinite chains. From the results of G. E. Shilov (^{1}) it follows that in these rings even a local theorem of Weierstrass type on approximation by polynomials cannot hold. In connection with this there arises the question of the validity of a similar theorem in arbitrary Banach spaces of infinitely differentiable functions with a norm of type (1) (see below). In the present note a negative answer to this question is given. The structure of the corresponding subspaces in which the Weierstrass theorem holds is also clarified.

2°. Let $\{A_n\}_0^\infty$ be a fixed increasing sequence of positive numbers, $A_0 = 1$. We introduce the following Banach spaces:

a) $D_{\{A_n\}}$ is the space of infinitely differentiable complex-valued functions $f(x)$ $(-\infty < x < \infty)$ with norm

$$\|f\| = \sup_{\substack{-\infty<x<\infty \\ n\geq 0}} \frac{|f^{(n)}(x)|}{A_n} < \infty; \tag{1}$$

b) $\widetilde{D}_{\{A_n\}}$ is the subspace of $D_{\{A_n\}}$ consisting of functions $f(x)$ periodic with period 2π.

On the basis of well-known inequalities of A. N. Kolmogorov (^{2}), relating the upper bounds of successive derivatives, the sequence $\{A_n\}$ may be considered logarithmically convex, i.e.

$$\frac{A_1}{A_0} \leq \frac{A_2}{A_1} \leq \frac{A_3}{A_2} \leq \cdots \tag{2}$$

We shall also assume that

$$\lim_{n\to\infty} \frac{A_n}{A_{n-1}} = \lim_{n\to\infty} \sqrt[n]{A_n} = \infty; \tag{3}$$

otherwise, as is easy to see, $\widetilde{D}_{\{A_n\}}$ is finite-dimensional and consists of trigonometric polynomials of bounded degree, while $D_{\{A_n\}}$ consists of functions of exponential type with bounded exponent.

Denote by B the set of all entire functions $p(x)$ of exponential type, bounded on the axis $-\infty < x < \infty$, and by $\widetilde{B}_n$ the set of all trigonometric polynomials

$$\tilde{p}(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

of arbitrary,

degree n. By the classical inequality of S. N. Bernstein and (3), $B \subset D$, $\widetilde{B} \subset \widetilde{D}$.* Obviously, $\widetilde{B} = B \cap \widetilde{D}$. Let $D^0, \widetilde{D}^0$ denote, respectively, the closures of B and $\widetilde{B}$ in D.

Lemma.

$$\widetilde{D}^0 = D^0 \cap \widetilde{D}. \tag{4}$$

Proof. Obviously, $\widetilde{D}^0 \subset D^0 \cap \widetilde{D}$. We shall prove that $D^0 \cap \widetilde{D} \subset \widetilde{D}^0$, i.e., that every periodic (with period 2π) function from D which can be approximated with arbitrary accuracy (in the norm of D) by functions from B is also approximated by trigonometric polynomials. Let $f(x) \in \widetilde{D}$, $p(x) \in B$, $\|f - p\| < \varepsilon$. Consider the functions

$$p_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} p(x + 2k\pi) \quad (n = 0, 1, 2, ...).$$

By the periodicity of $f(x)$ we have

$$\|f - p_n\| < \varepsilon \quad (n = 0, 1, 2, ...).$$

On the other hand, using the uniform boundedness and equicontinuity on $(-\infty, \infty)$ of the family of functions $\{p_n(x)\}$, as well as $\{p_n^{(m)}(x)\}$ (for any fixed m), one can extract a subsequence $\{p_{n_i}(x)\}$ converging, together with all its derivatives, to some function $\tilde{p}(x)$. Obviously, $\tilde{p}(x) \in B$ and $\|f - \tilde{p}\| \le \varepsilon$. Moreover, $\tilde{p}(x)$ is periodic, since

$$p_n(x + 2\pi) - p_n(x) = \frac{1}{n+1} \{p[x + 2(n+1)\pi] - p(x)\} \to 0 \quad (n \to \infty).$$

Therefore $\tilde{p}(x)$ is a trigonometric polynomial, as was required to prove.

3°. We proceed to the formulation of the main results.

Theorem 1. $\widetilde{D}^0$ is a proper subspace of $\widetilde{D}$; in other words, not every element $f \in \widetilde{D}$ can be approximated with arbitrary accuracy by a trigonometric polynomial. Similarly, D^0 is a proper subspace of D.

Theorem 2. The space $\widetilde{D}$ is nonseparable.

Theorem 3. In order that an element $f \in D$ belong to D^0, it is necessary and sufficient that either of the two equivalent conditions hold:

$$1) \quad \sup_{-\infty<x<\infty} |f^{(n)}(x)| = o(A_n) \quad (n \to \infty); \tag{5}$$

2) the translation operation $f_\tau(x) = f(x-\tau)$ is strongly continuous in D, i.e.

$$\|f_\tau - f\| \to 0 \quad (\tau \to 0). \tag{6}$$

These same conditions are necessary and sufficient in order that an element $f \in \widetilde{D}$ belong to $\widetilde{D}^0$.

4°. We first prove Theorem 3. Let $p(x)$ be any function of the class B. On the basis of S. N. Bernstein' s inequality, $|p^{(n)}(x)| \leq Ca^n$ $(-\infty < x < \infty;\ n = 0, 1, 2, ...)$, where C, a are positive constants. Taking (3) into account, we obtain from this that $p(x)$ satisfies (5). Since the set of elements $f \in D$ for which (5) holds is closed, the necessity of condition (5) is proved. We shall show that (5) implies (6). Suppose that for some $f \in D$ condition (5) is fulfilled and let $\varepsilon > 0$. There is an N such that

$$|f^{(n)}(x)| < \frac{\varepsilon}{2} A_n \quad (-\infty < x < \infty)$$

for $n > N$. There exists, further, a δ such that for $|\tau| < \delta$

$$|f^{(n)}(x+\tau) - f^{(n)}(x)| < \varepsilon A_n \quad (-\infty < x < \infty;\ n = 0, 1, ..., N).$$

Hence it follows that $\|f_\tau - f\| < \varepsilon$ $(|\tau| < \delta)$, i.e. (6).

Now let $f \in D$ satisfy condition (6); we shall prove that $f \in D^0$.

* Here and in what follows we omit the index $\{A_n\}$ for convenience.

Consider the following functions, which obviously belong to the class B:

$$P_n(x) = \frac{1}{\pi n} \int_{-\infty}^{\infty} \frac{\sin^2 n(x-\tau)}{(x-\tau)^2} f(\tau)\, d\tau = \frac{1}{\pi n} \int_{-\infty}^{\infty} f(x-\tau) \frac{\sin^2 n\tau}{\tau^2}\, d\tau =$$

$$= \frac{1}{\pi n} \int_{-\infty}^{\infty} f_\tau \frac{\sin^2 n\tau}{\tau^2}\, d\tau \quad (n = 1, 2, \ldots), \tag{7}$$

where the last integral is understood as an abstract one in the space D. We have

$$\|P_n - f\| \leq \frac{1}{\pi n} \int_{-\infty}^{\infty} \|f_\tau - f\| \frac{\sin^2 n\tau}{\tau^2}\, d\tau = \frac{1}{\pi n} \left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} \right) \|f_\tau - f\| \frac{\sin^2 n\tau}{\tau^2}\, d\tau \leq$$

$$\leq \frac{2\|f\|}{\pi n} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \frac{\sin^2 n\tau}{\tau^2}\, d\tau + \max_{|\tau| \leq \delta} \|f_\tau - f\| \leq \frac{4\|f\|}{\pi n \delta} + \max_{|\tau| \leq \delta} \|f_\tau - f\|.$$

Using (6), it is easy to obtain from the last estimate that $\|P_n - f\| \to 0$ $(n \to \infty)$. The last assertion of Theorem 3 follows from what has just been proved and from (4).

We pass to the proof of Theorem 1. Denote $A_{n-1}/A_n = \mu_n$ $(n = 1, 2, \ldots)$. By virtue of (2) and (3), μ_n tends monotonically to zero $(n \to \infty)$. Consider the functions

$$\chi_0(x) \equiv 1; \qquad \chi_n(x) = A_n \mu_n^n e^{ix/\mu_n} = \frac{\mu_n^n}{\mu_1 \mu_2 \cdots \mu_n} e^{ix/\mu_n} \quad (n = 1, 2, \ldots). \tag{8}$$

Let us note the following properties of χ_n:

a) $\left|\chi_n^{(m)}(x)\right|$ does not depend on x $(n, m = 0, 1, 2, \ldots)$;

b) $\left|\chi_n^{(m)}(x)\right| \leq A_m$ $(n, m = 0, 1, 2, \ldots)$;

c) $\left|\chi_n^{(n)}(x)\right| = A_n$ $(n = 0, 1, 2, \ldots)$;

d) $\left|\chi_n^{(m)}(x)\right| \to 0$ $(n \to \infty,\ m \text{ fixed})$;

e) $\left|\chi_n^{(m)}(x)\right| = o(A_m)$ $(m \to \infty,\ n \text{ fixed})$.

Using these properties, one can construct a sequence of integers

$$n_0 = N_0 = 0 < n_1 < N_1 < n_2 < N_2 < \cdots < N_{k-1} < n_k < N_k < \cdots$$

so that the following conditions are satisfied: for all n, $N_{k-1} \leq n < N_k$,

$$\left|\chi_{n_i}^{(n)}(x)\right| \leq \frac{A_n}{3^k} \quad (i < k); \tag{9}$$

$$\left|\chi_{n_i}^{(n)}(x)\right| \leq \frac{A_n}{4^{i-k}} \quad (i > k). \tag{10}$$

Indeed, suppose $n_1, N_1, n_2, N_2, \dots, n_{k-1}, N_{k-1}$ have already been chosen; using property e), choose N_k so that for $n \geq N_{k-1}$ (9) is fulfilled; after this, using d), choose n_k so that for $N_{i-1} \leq n < N_i$ $(i < k)$ the inequalities $\left|\chi_{n_k}^{(n)}(x)\right| \leq A_n/4^{k-i}$ are fulfilled. We now show that the function

$$\sigma(x) = \chi_{n_1}(x) + \chi_{n_2}(x) + \cdots \tag{11}$$

belongs to D, but does not belong to D^0. Let $N_{k-1} \leq n < N_k$, and let x be arbitrary. By virtue of (9) and (10), as well as properties b) and c), we have

$$\left|\sigma^{(n)}(x)\right| \leq \sum_{i=1}^{k-1} \left|\chi_{n_i}^{(n)}(x)\right| + \left|\chi_{n_k}^{(n)}(x)\right| + \sum_{i=k+1}^{\infty} \left|\chi_{n_i}^{(n)}(x)\right| \leq$$

$$\leq \frac{A_n}{3} + A_n + A_n \sum_{i=1}^{\infty} 4^{-i} = \frac{5}{3} A_n;$$

$$|\sigma^{(n_k)}(x)| \geq |\chi_{n_i}^{(n_k)}(x)| - \sum_{i=1}^{k-1} |\chi_{n_i}^{(n_k)}(x)| - \sum_{i=k+1}^{\infty} |\chi_{n_i}^{(n_k)}(x)| \geq$$

$$\geq A_{n_k} - \frac{A_{n_k}}{3} - A_{n_k} \sum_{i=1}^{\infty} 4^{-i} = \frac{1}{3} A_{n_k}.$$

Therefore, at every point x,

$$|\sigma^{(n)}(x)| \leq \frac{5}{3} A_n \quad (n = 0, 1, 2, \dots); \qquad \overline{\lim_{n\to\infty}} \frac{|\sigma^{(n)}(x)|}{A_n} \geq \frac{1}{3}. \tag{12}$$

On the basis of Theorem 3, $\sigma \in D$, but $\bar{\sigma} \notin D$. By modifying the construction somewhat, one can arrange that the functions $\chi_{n_i}(x)$ are periodic with period 2π and still satisfy (9) and (10). Then $\sigma \in \widetilde{D}$, but $\bar{\sigma} \notin \widetilde{D}^0$. Theorem 1 is proved.

Selecting from the sequence $\{n_k\}$ all possible subsequences $\{n_{k_\nu}\}$ and constructing the functions

$$\sigma_{\{k_\nu\}}(x) = \chi_{n_{k_1}}(x) + \chi_{n_{k_2}}(x) + \cdots, \tag{13}$$

one can, by repeating the preceding arguments, show that for two different subsequences $\{k_\nu\}$, $\{k'_\nu\}$,
$\|\sigma_{\{k_\nu\}} - \sigma_{\{k'_\nu\}}\| \geq 1/3$, provided only that the set $(\{k_\nu\} \cup \{k'_\nu\}) \setminus (\{k_\nu\} \cap \{k'_\nu\})$ is infinite. Thus there exists a continuum of elements $\sigma_{\{k_\nu\}} \in \widetilde{D}$ whose mutual distances are not less than $1/3$, which proves Theorem 2.

5°. To formulate the "local" Weierstrass theorem, introduce the concept of a norm on an interval (a, b):

$$\|f\|_{(a,b)} = \sup_{\substack{a<x<b \\ n=0,1,2,\ldots}} \frac{|f^{(n)}(x)|}{A_n}. \tag{14}$$

Theorem 4. *In order that a function $f(x) \in D$ be "locally approximable" by functions $p(x) \in B$, i.e., that for every finite interval (a, b) and every $\varepsilon > 0$ there exist a function $p(x) \in B$ such that $\|f - p\|_{(a-b)} < \varepsilon$, it is necessary and sufficient that*

$$\sup_{a<x<b} |f^{(n)}(x)| = o(A_n) \qquad (n \to \infty), \tag{15}$$

whatever the finite interval (a, b) may be.

To prove Theorem 4 it is necessary to consider the functions $P_n(x)$ (see (7)) and to refine somewhat the argument used in the proof of Theorem 3.

Theorem 4 remains valid if, in it, the class B is replaced by the set of finite trigonometric sums $\sum c_n e^{i\gamma_n x}$ (c_n arbitrary complex numbers, γ_n arbitrary real numbers). To prove this, one must suitably approximate the functions (7) by trigonometric sums, for example by B. M. Levitan' s polynomials ([3]).

Let us note that, by virtue of (12), the function $\sigma(x)$ cannot be approximated on any interval by functions of the class B.

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Received
10 I 1963

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Note: Figure translations are in progress. See original paper for figures.