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# MATHEMATICS

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Abstract

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## MATHEMATICS

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# ON THE UNIQUE DETERMINATION OF OVALOIDS WITH CUTS

(Presented by Academician I. N. Vekua, May 11, 1963)

Consider an ovaloid with cuts. The known theorem on the unique determination of ovaloids, as applied to ovaloids with cuts, may be formulated as follows: an ovaloid with cuts is uniquely determined under the condition

$$\Delta k^+(s) = \Delta k^-(s); \quad \Delta \varkappa^+(s) = \Delta \varkappa^-(s), \quad (*)$$

where  $\Delta k^\pm(s)$  and  $\Delta \varkappa^\pm(s)$  are, respectively, the increments of curvature and torsion of the left and right banks of the cut under an isometric transformation of the surface. Conditions (\*) mean the gluing of the ovaloid along the lines of the cut. In the present paper the unique determination of ovaloids with cuts is proved under boundary conditions considerably weaker than conditions (\*). In particular, it is established that an ovaloid with cuts along geodesics is uniquely determined under the condition

$$\Delta k^+(s) = \alpha^2(s)\Delta k^-(s); \quad \Delta \varkappa^+(s) = \beta^2(s)\Delta \varkappa^-(s),$$

where  $\alpha(s), \beta(s)$  are arbitrary smooth functions;  $\alpha(s) \neq 0$ ;  $\beta(s) \neq 0$ ;  $s$  is the arc length of the cut. At the same time, an integral representation is established for the complex function of bendings generated by an isometric transformation of an ovaloid with cuts.

The proofs are carried out by the methods of the book <sup>(1)</sup>.

No. 1. Let  $S$  be an ovaloid of positive Gaussian curvature  $K$  ( $K \geq k_0 > 0$ ) of class  $D_{3,p}$ ,  $p > 2$ . Further, let  $\mathcal{L}$  be a collection of closed or nonclosed nonintersecting contours of class  $C_\mu^1$ , lying on the ovaloid. The curve  $\mathcal{L}$  divides the neighborhoods of each of its points on  $S$  into two half-neighborhoods. Consider the curve  $\mathcal{L}$  in two copies:  $\mathcal{L}^+$  and  $\mathcal{L}^-$ . We shall regard the neighborhoods of the points of  $\mathcal{L}^+$  and  $\mathcal{L}^-$  as, respectively, the left and right half-neighborhoods of the points of  $\mathcal{L}$ . Under this condition the ovaloid  $S$  is transformed into a surface  $S_0$  with boundary  $\mathcal{L}^+ + \mathcal{L}^-$ , congruent to  $S$ . The operation of transforming the ovaloid  $S$  into the surface  $S_0$  will be called the **cutting of the ovaloid**

**along the curve**  $\mathcal{L}$ . We shall consider isometric transformations of ovaloids with cuts in the class  $D_{3,p}$ ,  $p > 2$ .

Introduce on the ovaloid  $S$  a unified isothermally conjugate parametrization  $u, v$  <sup>(1)</sup>, mapping the surface  $S$  homeomorphically onto the plane  $E$ . We shall assume that to the curves  $\mathcal{L}$  in the plane  $E$  there corresponds a collection of curves  $\Gamma$  situated in the finite part of the plane. Let the curves  $\mathcal{L}$  be the lines of cut of the ovaloid  $S$ . Then in the parametric plane the point  $M$ ,  $M \in \mathcal{L}$ , corresponds to a point  $t$ ,  $t \in \Gamma$ , which we shall also regard as a double point. The passage from one isothermally conjugate coordinate system  $u, v$  to another such coordinate system  $u_*, v_*$  is accomplished by a transformation of the form

$$u_* + iv_* = \psi(u + iv); \quad u_* + iv_* = \bar{\psi}(u + iv), \quad (1)$$

where  $\psi(z) = (\alpha z + \beta)/(\gamma z + \delta)$ ;  $\alpha\delta - \beta\gamma \neq 0$ ;  $z = u + iv$ . The fundamental forms of the surface have the form

$$ds^2 = g_{11}du^2 + 2g_{12}du dv + g_{22}dv^2; \quad II = b_0(du^2 + dv^2), \quad b_0 \neq 0;$$

the functions  $g_{ij}$  have, generally speaking, discontinuities on  $\Gamma$ .

Let the surface  $S$  with cuts be isometrically transformed into the surface  $S^*$  with coefficients  $b_{ij}(u, v)$  of the second fundamental form. The functions  $b_{ij}(u, v)$  in the domain  $D = E - \Gamma$  satisfy the fundamental equations of the theory of surfaces. Following the general idea of I. N. Vekua, we write these equations in complex form

$$\partial_{\bar{z}}w(z) - \overline{B(z)}\overline{w(z)} = [-C(z)P(z) + \partial_z P(z)]\sqrt{g(z)}\sqrt{K(z)};$$

$$P(z) = -\sqrt{K(z)} + \sqrt{K(z) + \frac{w(z)\overline{w(z)}}{g(z)\sqrt{K(z)}}}, \quad (2)$$

where

$$w = \frac{1}{2}\sqrt{K}(b_{11} - b_{22}) - i\sqrt{K}b_{12} \quad (3)$$

is the sought complex function of bendings;  $g = g_{11}g_{22} - g_{12}^2$ ;  $z = u + iv$ ;  $z \in D$ . The coefficients  $B(z)$  and  $C(z)$  are determined by formulas (6.52) and (6.54) of the book <sup>(1)</sup>, p. 128, and belong to the class  $L_{p,2}(D)$ ,  $p > 2$ .

**Fundamental lemma.** Let  $w(z)$  be a complex function of bendings of class  $D_{1,p}(D)$ ,  $p > 2$ . Suppose, further, that

$$a(z) = \begin{cases} -\bar{B} + \frac{(CP - \partial_z P)\sqrt{g}\sqrt{K}}{w}, & \text{if } w(z) \neq 0, \\ -\bar{B} + \frac{\partial_z w + \partial_z \bar{w}}{2\sqrt{gK}\sqrt{K}}, & \text{if } w(z) = 0. \end{cases}$$

In that case the function

$$\Phi(z) = w(z)e^{-\omega(z,w,w_z)}, \quad (4)$$

where

$$\omega(z) = \frac{1}{\pi} \iint_D \frac{a(\zeta) d\xi d\eta}{\zeta - z}, \quad \omega(z) \in C_\alpha(E), \quad \alpha = \frac{p-2}{p}, \quad p > 2, \quad \zeta = \xi + i\eta,$$

is holomorphic in  $D$ , and at infinity  $\Phi(z) = O(|z|^{-4})$ .

**Proof.** According to the results of the book <sup>(1)</sup>, in order to prove the lemma it is sufficient to show that

$$a(z) \in L_{p,2}(D), \quad p > 2, \quad \text{for } w(z) \in D_{1,p}(D), \quad p > 2.$$

For this purpose we estimate the behavior of  $w(z)$  as  $z \rightarrow \infty$ . According to (3) we have:

$$w(z) = -2\sqrt{K} (n_2 \Delta r_z + \Delta n_2 r_z + \Delta r_z \Delta n_z), \quad z \in D, \quad (5)$$

where  $r$  is the radius vector of the surface  $S$  mapped onto  $E$ ;  $n$  is the normal to the surface  $S$ ;  $\Delta r$  and  $\Delta n$  are the increments, respectively, of  $r$  and  $n$  under an isometric transformation of the surface  $S$ . The increments  $\Delta r$  and  $\Delta n$  may be regarded on  $S$  as continuously differentiable vector fields.

Let the isothermally conjugate coordinate net be transformed by formulas (1). Then we have the following transformation formulas:

$$n(z)_z = n(z_*)_{z_*} \psi'(z), \dots, \quad \Delta r(z)_z = \Delta r(z_*)_{z_*} \psi'(z). \quad (6)$$

Consequently, according to (5), (6), the equalities

$$w(z) = w(z_*) \psi'^2(z), \quad w(z) = O(|z|^{-4}) \quad (7)$$

hold in a neighborhood of  $z = \infty$ .

Since, as  $z \rightarrow \infty$ ,  $\sqrt{g} = O(|z|^{-4})$  and  $\sqrt{g(z)} = \sqrt{g(z_*)}|\psi'(z)|^2$  (see (1)), it follows from the second expression (2) that

$$P(z) = P(z_*); \quad P(z) = O(1), \quad z \rightarrow \infty. \quad (8)$$

Hence it follows that, in a neighborhood of infinity,

$$\partial_z P(z) = O(|z|^{-2}). \quad (9)$$

Consequently, by virtue of conditions (7), (8), (9),

$$a(z) = O(|z|^{-2}) \quad \text{as } z \rightarrow \infty. \quad (10)$$

The function  $a(z)$  also satisfies the relation

$$|a(z)| = |B| + \frac{|C||w|}{2\sqrt{K}} + \frac{|\partial_z \sqrt{K}||w|}{4K} + \frac{\left| \frac{\partial_z w}{\sqrt{g\sqrt{K}}} \right| + \left| \frac{\partial_z w}{\sqrt{g\sqrt{K}}} \right|}{2\sqrt{K}}. \quad (11)$$

From conditions (10) and (11) it follows that  $a(z) \in L_{p,2}(D)$ ,  $p > 2$ , if  $w \in D_{1,p}(D)$ ,  $p > 2$ . From condition (7) we obtain  $\Phi(z) = O(|z|^{-4})$  as  $z \rightarrow \infty$ . The lemma is proved.

**Remark.** The lemma holds for any surface of positive curvature with singularities (edges, conical points, cuts and punctures).

No. 2. Let  $S$  be an ovaloid with cuts  $\mathcal{L}$ . Denote by  $\Delta k^\pm$  and  $\Delta \tau_g^\pm$  the increments of curvature and geodesic torsion of  $\mathcal{L}^\pm$  under an isometric transformation of  $S$ .

**Theorem.** *An ovaloid  $S$  with cuts  $\mathcal{L}$  is uniquely determined under the conditions:*

$$\Delta k^+(M) = \alpha^2(M)\Delta k^-(M), \quad M \in \mathcal{L}; \quad \Delta \tau_g^+(M) = \beta^2(M)\Delta \tau_g^-(M), \quad M \in \mathcal{L}, \quad (12)$$

where  $\alpha, \beta$  are arbitrary functions of class  $C_\alpha(\mathcal{L})$ ,  $0 < \alpha < 1$ , satisfying the conditions:  $\alpha(M) \equiv \beta(M) \equiv 1$  in a neighborhood of the ends of  $\mathcal{L}$ ;  $\alpha(M) \neq 0$ ,  $\beta(M) \neq 0$ ,  $M \in \mathcal{L}$ .

**Proof.** Extend the cuts on the ovaloid so that the totality of the new cuts  $\tilde{\mathcal{L}}$ , together with the cuts  $\mathcal{L}$ , forms a finite number of closed nonintersecting contours of class  $C_\mu^1$ ,  $0 < \mu < 1$ . We shall require that, when the ovaloid with cuts  $\mathcal{L}_1 = \mathcal{L} + \tilde{\mathcal{L}}$  is bent, it “does not come apart” along the cuts  $\tilde{\mathcal{L}}$ , i.e., that on  $\tilde{\mathcal{L}}$  the relations

$$\Delta k^+(M) = \Delta k^-(M), \quad M \in \tilde{\mathcal{L}}; \quad \Delta \tau_g^+(M) = \Delta \tau_g^-(M), \quad M \in \tilde{\mathcal{L}} \quad (12')$$

hold.

Introduce on the surface  $S$  a single isometrically conjugate parametrization  $u, v$ , mapping the surface  $S$  homeomorphically onto the plane  $E$ . Without loss of generality, we shall assume that to the curves  $\mathcal{L}_1$  in the plane  $E$  there corresponds the totality of  $m$  nonintersecting contours

$$\Gamma = \sum_{i=1}^m \Gamma_i,$$

where  $\Gamma_1$  contains all the others inside it. As shown in (1),  $\Gamma \in C_\mu^1$ ,  $0 < \mu < 1$ . The  $m$ -connected domain lying inside  $\Gamma_1$  and outside the contours  $\Gamma_2, \dots, \Gamma_m$  will be denoted by  $\mathcal{D}^+$ . By  $\mathcal{D}^-$  we denote the complement of  $\mathcal{D}^+ + \Gamma$  in the complete plane. Passing in formulas (12) and (12') to the complex bending function  $w(z)$ , we obtain on  $\Gamma$  the boundary condition for  $w(z)$ :

$$w^+(t) = A(t; w^+; w^-) w^-(t) + B(t; w^+; w^-) \overline{w^-(t)}, \quad t \in \Gamma, \quad (13)$$

where  $A$  and  $B$  are certain functions of the arguments;  $A \in C_\alpha(\Gamma)$ ,  $0 < \alpha < 1$ ;  $B \in C_\alpha(\Gamma)$ ,  $0 < \alpha < 1$ , for  $w^\pm \in D_{1,p}(\mathcal{D}^\pm)$ ,  $p > 2$ ;  $w^\pm \equiv w$  for  $z \in \mathcal{D}^\pm$ .

It can be shown that for any  $\varphi^+(t)$  and  $\varphi^-(t)$  belonging to the class  $C_\alpha(\Gamma)$ ,  $0 < \alpha < 1$ , the coefficients  $A$  and  $B$  of the boundary condition (13) satisfy the conditions:

1.  $|A(t; \varphi^+(t); \varphi^-(t))| > |B(t; \varphi^+(t); \varphi^-(t))|, \quad t \in \Gamma.$
2.  $\text{Im} A(t; \varphi^+(t); \varphi^-(t)) = 0, \quad t \in \Gamma.$

To prove the unique determinacy of the surface  $S$ , it is necessary to establish the validity of the following proposition:

Every solution  $w$  of the boundary-value problem (2), (13), belonging to the class  $D_{1,p}(\mathcal{D}^\pm)$ ,  $p > 2$ , continuous in  $\mathcal{D}^\pm + \Gamma$  and satisfying condition (7), is identically equal to zero.

We reduce the boundary-value problem (2), (13), nonlinear with respect to  $w$ , to a problem in the class of analytic functions. For this we use the integral representation (4):

$$w^\pm(z) = \Phi^\pm(z) e^{\omega^\pm(z)}, \quad \text{if } z \in \mathcal{D}^\pm, \quad (14)$$

where  $\Phi^+$  and  $\Phi^-$  are holomorphic functions in  $\mathcal{D}^+$  and  $\mathcal{D}^-$ , respectively, and  $\omega^+$  and  $\omega^-$  are functions of the class  $C_\alpha(E)$ ,  $\alpha = (p-2)/p$ . Moreover,  $\Phi^\pm$  are continuous in  $\mathcal{D}^\pm + \Gamma$  and

$$\Phi^-(z) = O(|z|^{-4}) \quad \text{as } z \rightarrow \infty. \quad (15)$$

Substituting the expressions (14) into (13), we obtain

$$\Phi^+(t) = \alpha(t)\Phi^-(t) + \beta(t)\overline{\Phi^-(t)}, \quad t \in \Gamma, \quad (16)$$

where

$$\alpha = Ae^{\omega^- - \omega^+}, \quad \beta = Be^{\overline{\omega^-} - \omega^+}. \quad (17)$$

The coefficients  $\alpha(t)$  and  $\beta(t)$  of the boundary condition (16) depend on the unknown functions  $\Phi^+(t)$  and  $\Phi^-(t)$ . Equalities (17) give:

$$|\alpha| = |A| |e^{\omega^-}| |e^{-\omega^+}|; \quad |\beta| = |B| |e^{\omega^-}| |e^{-\omega^+}|.$$

Hence, by virtue of property (2), for the coefficients  $A$  and  $B$  it follows that

$$|\alpha(t)| > |\beta(t)|, \quad t \in \Gamma, \quad (18)$$

uniformly with respect to  $\Phi^+(t)$  and  $\Phi^-(t)$ .

Let us show that  $\text{Ind } \alpha(t) = 0$ . We have  $\text{Ind } \alpha(t) = \text{Ind } A + \text{Ind } e^{\omega^-} - \text{Ind } e^{\omega^+}$ . Since  $\omega^\pm \in C_\alpha(E)$ , it follows that  $\text{Ind } e^{\omega^+} = \text{Ind } e^{\omega^-} = 0$ . By property 2,  $\text{Ind } A = 0$  uniformly with respect to  $w^+$  and  $w^-$ ; consequently,

$$\text{Ind } \alpha(t) = 0 \quad \text{for arbitrary } \Phi^+(t) \text{ and } \Phi^-(t). \quad (19)$$

Let  $\Phi_0^\pm(z)$  be any solution of problem (16), continuous in  $\mathcal{D}^\pm + \Gamma$ , satisfying condition (7). We shall show that it is identically equal to zero. For this purpose, substitute the solution  $\Phi_0^\pm(z)$  into the boundary condition (16). The identity obtained may be regarded as a linear boundary condition for  $\Phi_0^+$  and  $\Phi_0^-$ , where  $\alpha(t)$  and  $\beta(t)$  are known functions satisfying conditions (18), (19). From the results of B. V. Boyarskii<sup>2</sup> and L. G. Mikhailov<sup>3</sup> it follows that, under condition (15), this boundary-value problem has only the zero solution. Consequently, the boundary-value problem (16) has the unique solution  $\Phi^\pm(z) \equiv 0$ , whence it follows that  $w^\pm(z) \equiv 0$ . The theorem is proved.

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## REFERENCES

<sup>1</sup> I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959. <sup>2</sup> B. V. Boyarskii, *Communications of the Academy of Sciences of the Georgian SSR*, **25**, No. 4 (1960). <sup>3</sup> L. G. Mikhailov, *DAN*, **139**, No. 2 (1961).

*Note: Figure translations are in progress. See original paper for figures.*

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