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Abstract

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ON THE STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR A PARABOLIC EQUATION WITH VARIABLE COEFFICIENTS

(Presented by Academician I. G. Petrovsky, 15 VI 1963)

In the present article it is shown how, from Nash' s estimates ([2]) for the fundamental solution itself of a parabolic equation and for its modulus of continuity, one can establish the fact of stabilization of a bounded solution of the Cauchy problem for which the initial function $u(x,0)=\varphi(x)$ has a "uniform mean" over the whole space x.

Consider the equation

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right),\quad x=(x_1,\dots,x_n)\in E^{(n)}\quad(0<t<\infty),\quad(1)$$

where $\|a_{ij}(x,t)\|$ is a symmetric matrix for which

$$c_1|\xi|^2\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le c_2|\xi|^2,\quad 0<c_1\le c_2.\quad(2)$$

Let $a_{ij}(x,t)$ and $\partial a_{ij}(x,t)/\partial x_i$ be continuous in t, with the continuity in t of the coefficients $a_{ij}(x,t)$ uniform with respect to $x\in E^{(n)}$. Suppose, furthermore, that $a_{ij}(x,t)$ and $\partial a_{ij}(x,t)/\partial x_i$ are bounded and satisfy the Hölder condition in x with exponent α_1, $0<\alpha_1\le 1$. Then it follows from the results of ([1],[2]) that every bounded solution $u(x,t)$ of equation (1), having second continuous derivatives with respect to x and a first continuous derivative with respect to t for $t>0$, is representable in the form

$$u(x,t)=\int_{E^{(n)}}Z(x,t,\xi,0)\varphi(\xi)\,d\xi,\quad(3)$$

where $\varphi(x)=u(x,0)$, and for the fundamental solution $Z(x,t,\xi,\tau)$ the estimates

$$Z(x,t,\xi,\tau) \le k_1(t-\tau)^{-n/2}\exp\left[-k_2\frac{|x-\xi|}{(t-\tau)^{1/2}}\ln\left(k_3\frac{|x-\xi|}{(t-\tau)^{1/2}}\right)\right]; \quad (4)$$

$$|Z(x,t,\xi_1,\tau)-Z(x,t,\xi_2,\tau)| \le k_4(t-\tau)^{-n/2}\left(\frac{|\xi_1-\xi_2|}{(t-\tau)^{1/2}}\right)^{\alpha}; \quad (5)$$

hold; the constants $\alpha, k_1, k_2, k_3, k_4$ depend only on n, c_1, c_2 from (2).

Theorem. Let $|\varphi(x)| \le K$,

$$\frac{1}{(2N)^n}\int_{|\gamma_i-x_i|\le N}\varphi(\gamma)\,d\gamma \underset{N\to\infty}{\longrightarrow} 0$$

uniformly with respect to $x \in E^{(n)}$.

Then

$$u(x,t) \underset{t\to\infty}{\longrightarrow} 0$$

uniformly with respect to $x \in E^{(n)}$.

Proof. In (3) make the substitution $\xi - x = t^{1/2}\beta$. We obtain

$$u(x,t) = t^{n/2}\int_{E^{(n)}} Z(x,t,x+t^{1/2}\beta,0)\,\varphi(x+t^{1/2}\beta)\,d\beta.$$

It follows from (4) that

$$t^{n/2}Z(x,t,x+t^{1/2}\beta,0) \le k_1\exp[-k_2|\beta|\ln(k_3|\beta|)] \quad (6)$$

uniformly in x and t.

In view of the estimate $|\varphi(x)| \le K$ and (6), for the given ε one can choose such a B that

$$u_1(x,t) = t^{n/2}\int_{|\beta_i|\ge B} Z(x,t,x+t^{1/2}\beta,0)\varphi(x+t^{1/2}\beta)\,d\beta \le \frac{\varepsilon}{2} \quad (7)$$

for all x and $t > 0$.

Let us show that by taking $t \ge T$, where T depends on $n, c_1, c_2, K, \varepsilon$, one can obtain

$$|u_2(x,t)| = t^{n/2}\left|\int_{|\beta_i|\le B} Z(x,t,x+t^{1/2}\beta,0)\varphi(x+t^{1/2}\beta)\,d\beta\right| \le \frac{\varepsilon}{2}.$$

Divide the domain $|\beta_i| \le B$ into $(2s)^n$ cubes by dividing B into s parts, where s, as will be seen from what follows, depends only on $n, c_1, c_2, K, \varepsilon$, and denote the ν-th cube by K_ν. Then

$$u_2(x,t) = \sum_{\nu=1}^{(2s)^n} t^{n/2}\int_{K_\nu} Z(x,t,x+t^{1/2}\beta,0)\varphi(x+t^{1/2}\beta)\,d\beta.$$

Decompose $\varphi(x+t^{1/2}\beta)$ into its positive and negative parts: $\varphi = \varphi^+ - \varphi^-$. Then

$$\begin{aligned}
u_2(x,t) = \sum_{\nu=1}^{(2s)^n} t^{n/2}\Bigg[&\int_{K_\nu} Z(x,t,x+t^{1/2}\beta,0)\varphi^+(x+t^{1/2}\beta)\,d\beta \\
&- \int_{K_\nu} Z(x,t,x+t^{1/2}\beta,0)\varphi^-(x+t^{1/2}\beta)\,d\beta\Bigg] \\
= \sum_{\nu=1}^{(2s)^n} t^{n/2}\Bigg[&Z(x,t,x+t^{1/2}\beta_{1,\nu,x,t},0)\int_{K_\nu}\varphi^+(x+t^{1/2}\beta)\,d\beta \\
&-Z(x,t,x+t^{1/2}\beta_{2,\nu,x,t},0)\int_{K_\nu}\varphi^-(x+t^{1/2}\beta)\,d\beta\Bigg] \\
= \sum_{\nu=1}^{(2s)^n} t^{n/2}\Big[&\{Z(x,t,x+t^{1/2}\beta_{1,\nu,x,t},0) - Z(x,t,x+t^{1/2}\beta_{2,\nu,x,t},0)\} \\
&\times \int_{K_\nu}\varphi^+(x+t^{1/2}\beta)\,d\beta + Z(x,t,x+t^{1/2}\beta_{2,\nu,x,t},0)\int_{K_\nu}\varphi(x+t^{1/2}\beta)\,d\beta\Big].
\end{aligned}$$

Since $\beta_{1,\nu,x,t}$ and $\beta_{2,\nu,x,t} \in K_\nu$, we have

$$|\beta_{1,\nu,x,t} - \beta_{2,\nu,x,t}| < \frac{B}{s}\sqrt{n}.$$

Let $\sigma_{\nu i}$ be the i-th coordinate of the center of the cube K_ν. Then K_ν is written as

$$|\beta_i - \sigma_{\nu i}| \le \frac{B}{2s}.$$

For a given δ, which depends on n, c_1, c_2, K and ε and which we shall define later, one can find such a T, depending on $n, c_1, c_2, K, \varepsilon$ and on the rate at which the mean of $\varphi(x)$ tends to zero, that

$$\frac{1}{(Bt^{1/2}/s)^n} \int_{|\gamma_i - x_i - \sigma_{\nu i} t^{1/2}| \leq Bt^{1/2}/2s} \varphi(\gamma)\, d\gamma \leq \delta \quad \text{for } t \geq T. \tag{8}$$

Make in the last integral the change of variables $\gamma_i = x_i + t^{1/2}\beta_i$. We obtain

$$\int_{|\beta_i - \sigma_{\nu i}| \leq B/2s} \varphi(x + t^{1/2}\beta)\, d\beta < \delta \left(\frac{B}{s}\right)^n \quad \text{for } t \geq T.$$

This means that

$$\int_{K_\nu} \varphi\left(x + t^{1/2}\beta\right)\, d\beta < \delta V_{K_\nu}$$

for $t \geq T$, where V_{K_ν} is the volume of the cube K_ν.

Let us estimate $u_2(x,t)$:

$$|u_2(x,t)| \leq \sum_{\nu=1}^{(2s)^n} \left[k_4 \left|\beta_{1,\nu,x,t} - \beta_{2,\nu,x,t}\right|^\alpha K V_{K_\nu} + M\delta V_{K_\nu}\right] \leq$$

$$\leq \left(k_4 \frac{B^\alpha}{s^\alpha} n^{\alpha/2} K + M\delta\right) \sum_{\nu=1}^{(2s)^n} V_{K_\nu} =$$

$$= \left(k_5 K \frac{B^\alpha}{s^\alpha} + M\delta\right)(2B)^n, \quad \text{where } M = \max_\beta \{k_1 \exp[-k_2|\beta| \ln(k_3|\beta|)]\},$$

and therefore M depends only on n, c_1, and c_2.

We now choose s and δ so that

$$k_5 K \frac{B^\alpha}{s^\alpha}(2B)^n \leq \frac{\varepsilon}{4}, \qquad M\delta(2B)^n \leq \frac{\varepsilon}{4}. \tag{9}$$

Thus, for $t > T$, where T depends on $n, c_1, c_2, K, \varepsilon$ and on the rate of convergence of the mean of $\varphi(x)$ to zero, we obtain

$$|u(x,t)| \leq |u_1(x,t)| + |u_2(x,t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

simultaneously for all $x \in E^{(n)}$, as was required to prove.

Remark. It can be shown that, in order to satisfy conditions (7) and (9), it is sufficient to take

$$B = k_6 \ln \frac{K k_7}{\varepsilon}, \quad s = \frac{E}{\varepsilon^{1/\alpha'}}, \quad \delta = F \varepsilon^{1+\alpha''},$$

where $0 < \alpha' < \alpha$, $\alpha'' > 0$; the constants k_6 and k_7 depend on n, c_1, c_2, the constant E depends on n, c_1, c_2, K, α', and the constant F depends on n, c_1, c_2, K, α''.

Therefore inequality (8) can be written as follows:

$$\frac{1}{(2D)^n} \int_{|\gamma_i - x_i - a_i| \le D} \varphi(\gamma)\, d\gamma \le \varepsilon F^{1+\alpha''}, \tag{10}$$

where

$$D \le |a_i| \le \frac{2E}{\varepsilon^{1/\alpha'}} D.$$

It is easy to show that (10) will hold for $D \ge D_0(\varepsilon, X)$, for all $|x_i| \le X$, already when the initial function $\varphi(x)$ has only angular means (see (3)).

Thus the following assertion holds:

If $|\varphi(x)| \le K$,

$$\frac{1}{A} \int_{\text{Rea}} \varphi(\xi)\, d\xi \longrightarrow 0 \quad \text{(notation as in (3))}, \quad \text{as } a_1 \to \infty, \dots, a_n \to \infty,$$

then

$$u(x,t) \longrightarrow 0 \quad \text{as } t \to \infty$$

uniformly in x, $|x_i| \le X$.

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Note added in proof. Estimate (7) is easily obtained without (4) from the estimate of the moment of the fundamental solution ([2]), which in our case has the form

$$\int |x - \xi|\, |Z(x,t,\xi,0)|\, d\xi \le L t^{1/2}.$$

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Note: Figure translations are in progress. See original paper for figures.