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Abstract

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ON FUNDAMENTAL SOLUTIONS OF ELLIPTIC EQUATIONS

(Presented by Academician S. L. Sobolev on 26 III 1963)

Let the operator

$$Lu \equiv \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + (c(x) + \lambda^2 g(x))u \qquad (1)$$

be given and elliptic in the whole N-dimensional space E_N, i.e. $a_{ij}(x) = a_{ji}(x)$,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{N}\xi_i^2 \quad (a = \text{const} > 0)$$

for arbitrary real $\xi_1, \dots, \xi_N$ at any point $x \in E_N$. We assume that: a) $a_{ij}(x), b_i(x) \in C^{(1,\mu)}(E_N)$; $c(x), g(x) \in C^{(0,\mu)}(E_N)$ $(\mu > 0)$; b) $g(x) \geq g_0 > 0$; c) outside a prescribed bounded domain T_0, $a_{ij}(x) \equiv \delta_{ij}$, $b(x), c(x) \equiv 0$, $g(x) \equiv 1$; d) the numerical parameter λ takes arbitrary complex values.

The note is devoted to two problems:

A. In the case $b_i(x) \equiv 0$, for all λ in a certain neighborhood of the upper λ-half-plane, the so-called **principal fundamental solution** of the self-adjoint operator L is constructed—an analogue of the solution $\exp(i\lambda|x-y|)/4\pi|x-y|$ of the three-dimensional equation $\Delta u + \lambda^2 u = 0$.

B. In the case of real coefficients ($\lambda = 0$), the existence of a fundamental solution of a general non-self-adjoint operator L in **any bounded domain** is proved.

The principal fundamental solution is known for the three-dimensional Schrödinger equation $\Delta u + (q(x) + \lambda^2)u = 0$ (see, for example, ([1])). With regard to problem B, the last result belongs to Yu. I. Lyubich ([2]), who proved the existence of a fundamental solution under more stringent conditions $a_{ij} \in C^{(4)}$, $b_i \in C^{(3)}$, $c \in C^{(2)}$. We note that below no assumptions whatever are made on the sign of the coefficient $c(x)$.

1. We shall need twice the following two basic propositions.

Theorem of E. M. Landis ([3]). *Let λ^2 be real; suppose, furthermore, that a function $v(x)$ belongs to the class $C^{(1)}$ in a bounded closed domain Ω with smooth boundary Σ, satisfies in Ω the equation $Lv = 0$, and on the boundary*

$$v\big|_\Sigma = \frac{\partial v}{\partial n}\bigg|_\Sigma = 0 \quad \left(\frac{\partial}{\partial n} \text{ is the derivative along the normal to } \Sigma\right).$$

Then $v(x) \equiv 0$ in Ω.

In ([3]) this theorem is proved in a different formulation. However, the changes in the proof adapting the theorem to conditions a) are trivial; they were indicated to us by E. M. Landis.

Lemma 1. *There exists a sufficiently small $\rho > 0$ such that in every ball Ω_ρ of radius ρ the following assertion is true. Let a function $\chi(x)$ of class $C^{(0,\mu)}(\Omega_\rho + \Sigma_\rho)$ (Σ_ρ is the boundary of Ω_ρ) be equal to zero on Σ_ρ and satisfy the equality*

$$\int_{\Omega_\rho} \chi(x) Lu(x)\, dx = 0$$

for every finite in Ω_ρ function $u(x)$ of class $C^{(2)}(\Omega_\rho)$. Then $\chi(x) \equiv 0$ in Ω_ρ.

2. We turn to the construction of the principal fundamental solution.

Definition 1. An S-kernel is a function $A(x, y)$ of a pair of points $x, y \in E_N$, satisfying the following requirements:
a) $A(x, y)$ depends analytically on the parameter λ for all $\lambda \neq 0$;
b) $A(x, y)$ is a Levi function (see (4)) for $x, y \in E_N$;
c) for any $y \in E_N$, $L_{xA}(x, y) = 0$, when $x \in E_N - T_0$, and

$$L_{x'} A(x', y) - L_{x''} A(x'', y) = O\left(|x' - x''|^\mu \rho^{\nu-\mu-N}\right) \qquad (\mu \leq \nu)$$

uniformly for $x', x'', y \in E_N$ (ρ is the distance from the point y to the segment $x'x''$);
d) for $\arg\lambda = 0$ the Sommerfeld condition is satisfied

$$\frac{\partial A}{\partial r_{xy}} - i\lambda A = o\left(r_{xy}^{(1-N)/2}\right). \tag{2}$$

Definition 2. A principal fundamental solution of the operator L is an S-kernel $G(x, y)$ for which $L_{xG}(x, y) \equiv 0$.

It is easy to give an example showing that the class of S-kernels is nonempty. Following E. E. Levi (5), represent G by means of an arbitrary S-kernel $A(x, y)$ in the form

$$G(x,y) = A(x,y) + \int_T A(x,t)R(t,y)\,dt,$$

where T is a ball containing the domain T_0. The function R is found from an integral equation which defines it as the resolvent of the kernel $L_{xA}(x,y)$. The function R exists for sufficiently large negative λ^2 in modulus; therefore it exists everywhere except at points that are isolated poles.

It can be shown that if λ is a pole of G, then there exist solutions of the equation

$$\varphi(y) = \int_T \overline{L_{xA}(x,y)}\,\varphi(x)\,dx \tag{3}$$

which are the same for any S-kernel A. Hence, with the aid of Lemma 1, we find the following properties of these solutions:
a) $\varphi(x) \in C^{(2)}(E_N)$;
b) $L\varphi(x) \equiv 0,\ x \in E_N$;
c) $\partial\varphi/\partial r - i\lambda\varphi = O\left(r^{-(1+N)/2}\right)$ $(r = |x|)$.

For $\operatorname{Im}\lambda \geq 0$, it follows from b), c) that $\varphi \in L_2(E_N)$. Since the operator L is self-adjoint, the functions φ can be nonzero only when $\operatorname{Im}\lambda^2 = 0$.

If $\lambda^2 \leq -\max c(x)/\min g(x)$, then $\varphi \equiv 0$ by the maximum principle. If $\lambda^2 > 0$, then, as A. Ya. Povzner (6) showed, from $\varphi \in L_2(E_N)$ and the equality $\Delta\varphi + \lambda^2\varphi = 0$, which holds outside T_0, it follows that $\varphi(x) = 0$ outside T_0. By Landis' theorem, $\varphi(x) \equiv 0$.

Thus, we have shown that the principal fundamental solution exists in the entire λ-plane, with the exception of the interval $\left(+\operatorname{Re}\sqrt{\max c(x)/\min g(x)}, 0\right)$ of the imaginary axis, where there may be at most a finite number of poles of the function $G(x,y;\lambda)$.

3. For $N = 2$ the operator L (with $g(x) \equiv 1$) is reduced to the form

$$Lu = \Delta u + a_1(x)\frac{\partial u}{\partial x_1} + a_2(x)\frac{\partial u}{\partial x_2} + \left(c(x) + \lambda^2\right)u. \tag{1'}$$

By a method similar to that used by L. D. Faddeev in (1) for estimates, for large $|\lambda|$, of the resolvent kernel of the three-dimensional Schrödinger operator, we found for the operator $(1')$ $(|\lambda| > \Lambda)$:

$$G(x,y) = -\frac{i}{\sqrt{2\pi}}H_0^1(r_{xy}\lambda)\left(1 + \lambda^{-1/2}g(x,y;\lambda)\right),$$

$$\frac{\partial}{\partial x_i}G(x,y) = -\frac{i\lambda(x_i - y_i)}{\sqrt{2\pi}\,r_{xy}}H_1^1(r_{xy}\lambda)\left(1 + \lambda^{-1/2}g'(x,y;\lambda)\right), \tag{4}$$

where g, g' are uniformly bounded for all x, y, λ and are zero for $x = y$. Here we require that $a_i \in C^{(2)}$, $c \in C^{(1)}$.

4. We turn to the question of the existence of a fundamental solution in a bounded domain Ω ($\lambda = 0$). As is known ((4), pp. 221-222), the question reduces to the existence in Ω of a solution of the equation $Lu = \varphi$ for an arbitrary function $\varphi \in C^{(0,\mu)}$. Adding the boundary condition $u|_\Sigma = \zeta$, we reduce the resulting Dirichlet problem to an equivalent system of integral equations. In doing so one uses the principal fundamental solution of the equation

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - u = 0.$$

The boundary-value problem is solvable when conditions of the form

$$\int_\Omega v_k \varphi\, dx + \int_\Sigma \omega_k \zeta\, d\sigma = 0 \qquad (k = 1, \ldots, q) \tag{5}$$

are satisfied, or

$$\int_\Sigma \omega_k \zeta\, d\sigma = \alpha_k. \tag{5'}$$

We prove, with the aid of Lemma 1 and Landis' theorem, that the functions ω_k are linearly independent; in this case the requirements $(5')$ are easily satisfied. Choosing ζ in an appropriate way, we solve the boundary-value problem, and hence also the equation $Lu = \varphi$. This proves the existence of a fundamental solution.

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Note: Figure translations are in progress. See original paper for figures.