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Abstract

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SOME INEQUALITIES BETWEEN NORMS OF PARTIAL DERIVATIVES OF FUNCTIONS OF MANY VARIABLES

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Let $f(x_1, \dots, x_n)$ be a continuous function given in a domain Ω of the n-dimensional Euclidean space E^n of points $\mathbf{x} = (x_1, \dots, x_n)$, having continuous derivatives of arbitrary order. Let $n+1$ integral nonnegative vectors be given,

$$\mathbf{r}_i = (l_1^i, \dots, l_n^i) \quad (i = 0, 1, \dots, n) \quad (l_j^i \geqslant 0 \text{ integers}).$$

The problem is to find integral nonnegative vectors $\vec{\rho} = (\nu_1, \dots, \nu_n)$ and numbers $q \geqslant p \geqslant 1$ for which the inequality

$$\|D^{\vec{\rho}} f\|_{L_q(\Omega)} \leqslant C \sum_{i=0}^{n} \|D^{\mathbf{r}_i} f\|_{L_p(\Omega)} \tag{1}$$

holds, where C is a constant independent of f,

$$D^{\mathbf{r}} f = \frac{\partial^{l_1}}{\partial x_1^{l_1}} \cdots \frac{\partial^{l_n} f}{\partial x_n^{l_n}} \quad (\mathbf{r} = (l_1, \dots, l_n)).$$

The results of the present note are a development of the results of note ([1]) on this question. Some of the inequalities given below may be regarded as a generalization, in a certain direction, of the known inequalities of S. L. Sobolev ([2]) and S. M. Nikol' skii ([3]); some new inequalities are also given.

I. We shall say that a domain D of the space E^n belongs to the class $C(\mathcal{H}^k)$, where $1 \leqslant k \leqslant n$, if for every point $\mathbf{x} \in D$ there exists a k-dimensional cube with vertex at $\mathbf{x}$ and with edges of length $\mathcal{H}$, parallel to the coordinate axes $x_{n-k+1}, \dots, x_n$, contained in Ω. We shall further say that S^m, an m-dimensional surface of the space E^n, is a surface of class $C^{(1)}$ if it is given by the equations $x_1 = x_1, \dots, x_m = x_m$, $x_{m+1} = \varphi_{m+1}(x_1, \dots, x_m), \dots, x_n = \varphi_n(x_1, \dots, x_m)$, where the functions $\varphi_i(x_1, \dots, x_m)$ are defined in some domain Ω^m of the space E^m of points $\mathbf{x}^m = (x_1, \dots, x_m)$ and have continuous bounded partial derivatives of first order in Ω^m.

II. Let natural numbers $s \leqslant n$ and n_i $(i = 1, \dots, s)$ be given, for which $n_1 + \dots + n_s = n$.

Consider the following table of numbers, consisting of $s+1$ rows (the rule for forming the numbers is clear from the table):

i	
0	1
1	$1,\ n_1$
2	$1,\ n_1,\ n_2,\ n_1 n_2$
3	$1,\ n_1,\ n_2,\ n_3,\ n_1 n_2,\ n_1 n_3,\ n_2 n_3,\ n_1 n_2 n_3$
.	. . . .
.	. . . .
s	$1,\ n_1,\ n_2, \dots, n_s,\ n_1 n_2, \dots, n_1 n_2 \dots n_s$

(2)

By α_i we denote an arbitrary number taken from the i-th row; α_i is either equal to 1 (the first number of the row), or is a number of the form $\alpha_i = n_{i_1} \cdots n_{i_{k_i}}$, where k_i denotes the number of factors in α_i, and the indices $i_1, \dots, i_{k_i}$ indicate exactly which factors enter into α_i ($i_1, \dots, i_{k_i}$ are certain natural numbers not exceeding i). To the number α_i we associate α_i vectors, for which we introduce the notation: $\mathbf{r}_{i;0}$, if α_i is equal to the first number of the i-th row ($\alpha_i = 1$); $\mathbf{r}_{i;j_{i_1},\dots,j_{i_{k_i}}}$, ($j_{i_t} = 1, \dots, n_{i_t};\ t = 1, \dots, k_i$), if $\alpha_i = n_{i_1} \cdots n_{i_{k_i}}$.

The coordinates of an arbitrary vector $\mathbf{r}$ will be denoted by $r_{\lambda,\mu}$ ($\lambda = 1, \dots, s;\ \mu = 1, \dots, n_\lambda$), i.e.

$$\mathbf{r} = (l_{1,1}, \dots, l_{1,n_1}; \dots; l_{s,1}, \dots, l_{s,n_s});$$

in particular, the coordinates of $\mathbf{r}_{i;j_{i_1},\dots,j_{i_{k_i}}}$ are denoted by

$$l_{\lambda,\mu}^{i;j_{i_1},\dots,j_{i_{k_i}}}.$$

We now state a somewhat more general theorem, from which, in particular, sufficient conditions for the validity of inequality (1) for $q = p$ will follow.

Theorem 1. Let natural numbers $s \le n$ and n_i ($i = 1, \dots, s$) be given, for which

$$n_1 + \dots + n_s = n,$$

and the table (2) corresponding to them, as well as numbers α_i ($i = 0, 1, \dots, s$),

$$\sum_{i=0}^{s} \alpha_i = N.$$

Suppose, further, that N integral nonnegative vectors $\mathbf{r}_{i;j_{i_1},\dots,j_{i_{k_i}}}$ (or $\mathbf{r}_{i;0}$), corresponding to the numbers α_i, and an integral nonnegative vector

$$\vec{\rho} = (v_{1,1}, \dots, v_{s,n_s})$$

are given, whose coordinates satisfy the conditions:

1.

$$\mathbf{r}_{i;0}: \quad l^{i;0}_{\lambda,\mu} = v_{\lambda,\mu} \quad (\lambda = 1, \dots, i,\ i+2, \dots, s;\ \mu = 1, \dots, n_\lambda),$$

$$l^{i;0}_{i+1,\mu} \le v_{i+1,\mu} \quad (\mu = 1, \dots, n_{i+1});$$

$$\mathbf{r}_{i;j_{i_1},\dots,j_{i_{k_i}}}: \quad l^{i;j_{i_1},\dots,j_{i_{k_i}}}_{\lambda,\mu} = v_{\lambda,\mu} \quad \text{for } \lambda = 1, \dots, s, \text{ but } \lambda \ne i_1, \dots, \lambda \ne i_{k_i},$$

$$\lambda \ne i+1, \quad \mu = 1, \dots, n_\lambda;$$

$$l^{i;j_{i_1},\dots,j_{i_{k_i}}}_{i_t,j_{i_t}} > v_{i_t,j_{i_t}} \quad (t = 1, \dots, k_i);$$

$$l^{i;j_{i_1},\dots,j_{i_{k_i}}}_{i_t,\mu} \le v_{i_t,\mu} \quad (\mu = 1, \dots, n_{i_t},\ \mu \ne j_{i_t};\ t = 1, \dots, k_i);$$

$$l^{i;j_{i_1},\dots,j_{i_{k_i}}}_{i+1,\mu} \le v_{i+1,\mu} \quad (\mu = 1, \dots, n_{i+1}).$$

2. For all $\lambda = i_1, \dots, i_{k_i}$ $(i = 1, \dots, s)$ and $\mu = 1, \dots, n_\lambda$ there exist numbers $x_{\lambda,\mu} > 0$ such that

$$\sum_{\mu=1}^{n_{i_t}} v_{i_t,\mu} x_{i_t,\mu} < \sum_{\mu=1}^{n_{i_t}} l^{i;j_{i_1},\dots,j_{i_{k_i}}}_{i_t,\mu} x_{i_t,\mu} \quad (t = 1, \dots, k_i)$$

for all vectors $\mathbf{r}_{i;j_{i_1},\dots,j_{i_{k_i}}}$

$$(j_{i_t} = 1, \dots, n_{i_t},\ t = 1, \dots, k_i,\ i = 1, \dots, s).$$

Assume that

$$D^{r_i} f \in L_p(\Omega) \quad (i = 1, \dots, N) \quad (p \ge 1),$$

where $\mathbf{r}_i$ are the newly renumbered given N vectors, $\Omega \in C(\mathcal{H}^n)$. Then the inequality

$$\|D^{\vec{\rho}} f\|_{L_p(\Omega)} \le C \sum_{i=1}^{N} \|D^{r_i} f\|_{L_p(\Omega)}, \tag{3}$$

holds, where C is a constant independent of f.

From inequality (3) there follows inequality (1) in the case when

$$N = \sum_{i=0}^{s} \alpha_i \le n + 1.$$

In particular, for $n_1 = \dots = n_{s-1} = 1,\ n_s = n + 1 - s$,

$$\sum_{i=0}^{s} \alpha_i$$

will not exceed $n+1$ for any choice of α_i. To this case there corresponds the following simpler theorem.

Theorem $1'$**.** Let a natural number $s \leq n$ and integer nonnegative vectors $\mathbf{r}_i$ $(i = 0, 1, \dots, n)$ and $\vec{\rho} = (v_1, \dots, v_n)$ be given, whose coordinates satisfy the conditions:

1. $\mathbf{r}_i$ $(i = 0, 1, \dots, s-2)$:

$$l_j^i \geq v_j \quad (j = 1, \dots, i), \qquad l_{i+1}^i \leq v_{i+1}, \qquad l_j^i = v_j \quad (j = i+2, \dots, n);$$

$$\mathbf{r}_{s-1}: \quad l_j^i \geq v_j \quad (j = 1, \dots, s-1), \qquad l_j^i \leq v_j \quad (j = s, \dots, n); \tag{A}$$

$\mathbf{r}_i$ $(i = s, \dots, n)$: for each fixed $j = 1, \dots, s-1$, either

$$l_j^i > v_j \quad (i = s, \dots, n),$$

or

$$l_j^i = v_j \quad (i = s, \dots, n),$$

and either a)

$$l_i^i > v_i \quad (i = s, \dots, n), \qquad l_j^i \leq v_i \quad (j = s, \dots, n;\ j \neq i;\ i = s, \dots, n),$$

or b)

$$l_j^i = v_j \quad (i = s, \dots, n;\ j = s, \dots, n).$$

2. If the vectors $\mathbf{r}_i$ $(i = s, \dots, n)$ satisfy conditions a), then let there exist numbers $\chi_i > 0$ $(i = s, \dots, n)$ such that

$$\sum_{j=s}^{n} v_j \chi_j < \sum_{j=s}^{n} l_j^i \chi_j \quad (i = s, \dots, n).$$

Let, furthermore, $D^{\mathbf{r}_i} f \in L_p(\Omega)$ $(p \geq 1)$, $\Omega \in C(\mathcal{H}^n)$. Then

$$\left\| D^{\vec{\rho}} f \right\|_{L_p(\Omega)} \leq C \sum_{i=0}^{n} \left\| D^{\mathbf{r}_i} f \right\|_{L_p(\Omega)}, \tag{4}$$

where C is a constant independent of f.

For a concrete choice of the parameters in Theorem $1'$ one obtains various inequalities. For example, for $s = n$, $l_i > v_i$, $0 \leq k_i \leq v_i$ $(i = 1, \dots, n)$, we obtain the inequality

$$\left\| D^{v_1 + \dots + v_n} f \right\|_{L_p(\Omega)} \leq$$

$$\leq C \left(\left\| D^{l_1 + \dots + l_n} f \right\|_{L_p(\Omega)} + \sum_{i=1}^{n} \left\| D^{v_1 + \dots + v_{i-1} + k_i + v_{i+2} + \dots + v_n} f \right\|_{L_p(\Omega)} \right).$$

III. From the results of work [1] it follows that inequality (1) for an arbitrary domain of class $C(\mathcal{H}^n)$ for $q > p$ is possible only when the coordinates of the vectors $\mathbf{r}_i$ $(i = 0, 1, \dots, n)$ and $\vec{\rho}$ satisfy the conditions:

$$l_j^0 \le v_j \quad (j = 1, \dots, n) \qquad \text{for } i = 0,$$
$$l_j^i \le v_j \quad (j = 1, \dots, i-1, i+1, \dots, n), \qquad l_i^i > v_i \quad \text{for } i = 1, \dots, n, \qquad (\ )$$

which are obtained from conditions (A) for $s = 1$. Under these conditions the usual embedding theorems hold.

Theorem 2. Let integer nonnegative vectors $\mathbf{r}_i$ $(i = 0, 1, \dots, n)$ and $\vec{\rho}$ be given, for which conditions ( ) are valid, and, in addition, let there exist numbers $\chi_j > 0$ $(j = 1, \dots, n)$ such that

$$F_1 = \sum_{j=1}^{n} v_j \chi_j < \sum_{j=1}^{n} l_j^i \chi_j = F \quad (i = 1, \dots, n). \tag{5}$$

Let, further:

1. A natural number m and numbers p and q are given, satisfying the inequalities:
$$1 \le m \le n, \quad 1 \le p \le q \le \infty, \quad F - F_1 - \frac{1}{p}\sum_{j=1}^{n} \varkappa_j + \frac{1}{q}\sum_{j=1}^{m} \varkappa_j = \varepsilon_m \ge 0.$$
2. $D^{r_i} f \in L_p(\Omega)$ $(i = 0, 1, \dots, n)$, $\Omega \in C(\overline{\mathfrak{H}^n})$.
3. S^m is an m-dimensional surface of class $C^{(1)}$, contained in $\overline{\Omega}$ $(S^n \equiv \Omega)$.

Then, under one of the following conditions: a) $\varepsilon_m > 0$, b) $\varepsilon_m = 0$, $1 < p < q < \infty$, the inequality

$$\left\|D^{\vec{\rho}} f\right\|_{L_q(S^m)} \le C_1 h^{-\delta_m} \left\|D^{r_0} f\right\|_{L_p(\Omega)} + C_2 h^{\varepsilon_m} \sum_{i=1}^{n} \left\|D^{r_i} f\right\|_{L_p(\Omega)}$$

holds, where

$$\delta_m = F - \varepsilon_m - \sum_{j=1}^{n} l_j^0 \varkappa_j, \quad 0 < h \le \mathfrak{H}^{1/\varkappa_j} \ (j = 1, \dots, n),$$

and C_1 and C_2 are constants independent of f and h.

Remark. If the vectors $\mathbf{r}_i$ and $\vec{\rho}$ satisfy conditions (B), but among the coefficients $\varkappa_i$ satisfying (5) there are also negative ones, then inequality (1) is possible only for $q = p$ and only in the case when $k + 1$ vectors $\mathbf{r}_i$ $(1 \le k < n)$ and the vector $\vec{\rho}$ satisfy conditions 1-2 of the following theorem.

Theorem 3. Let the integer nonnegative vectors $\mathbf{r}_i$ $(i = 0, n-k+1, \dots, n)$ and $\vec{\rho}$ satisfy the conditions:

1.
$$l_j^0 = v_j \ (j = 1, \dots, n-k), \quad l_j^0 \le v_j \ (j = n-k+1, \dots, n);$$
$$l_j^i = v_j \ (j = 1, \dots, n-k), \quad l_i^i > v_i,$$
$$l_j^i \le v_j \ (j = n-k+1, \dots, n; \ j \ne i) \quad \text{for } i = n-k+1, \dots, n.$$

2. There exist numbers $\varkappa_j > 0$ $(j = n-k+1, \dots, n)$ such that
$$F_1 = \sum_{j=n-k+1}^{n} v_j \varkappa_j < \sum_{j=n-k+1}^{n} l_j^i \varkappa_j = F \quad (i = n-k+1, \dots, n).$$

Let, further:

3.
$$m \ge n-k, \quad p \ge 1, \quad F - F_1 - \frac{1}{p} \sum_{j=n-m+1}^{n} \varkappa_j > 0.$$

4. $D^{r_i} f \in L_p(\Omega)$ $(i = 0, n-k+1, \dots, n)$, $\Omega \in C(\overline{\mathfrak{H}^k})$.

5. S^m is an m-dimensional surface of class $C^{(1)}$, contained in $\overline{\Omega}$ $(S^n \equiv \Omega)$.

Then
$$\left\| D^{\vec{\rho}} f \right\|_{L_p(S^m)} \le C \left(\left\| D^{r_0} f \right\|_{L_p(\Omega)} + \sum_{i=n-k+1}^{n} \left\| D^{r_i} f \right\|_{L_p(\Omega)} \right),$$

where C does not depend on f.

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Note: Figure translations are in progress. See original paper for figures.