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Abstract

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ON CONSTRUCTIVE MAPPINGS OF POLYHEDRA

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In the present communication we use the notions of a constructive metric space, of a constructive operator from one constructive metric space to another, of a continuous constructive operator, and of a uniformly continuous constructive operator. Definitions of these notions may be found in ([1]). Sets specified by means of lists will be called **finite sets**.

A **constructive locally finite simplicial complex** will mean any list of the form $(A, \mathfrak{M}, \mathfrak{N}, \mathfrak{A})$, where:

1) A is an alphabet containing the alphabet of constructive real numbers*;
2) $\mathfrak{M}$ is an enumerable set of words in the alphabet A;
3) $\mathfrak{N}$ is an enumerable family consisting of certain nonempty finite subsets of $\mathfrak{M}$ and satisfying the following conditions: a) every nonempty subset of each element of the family $\mathfrak{N}$ belongs to $\mathfrak{N}$, and b) for any word P belonging to $\mathfrak{M}$, the set of those elements of the family $\mathfrak{N}$ which contain P is nonempty and finite;
4) $\mathfrak{A}$ is an algorithm transforming any element of the set $\mathfrak{M}$ into the list of all elements of the family $\mathfrak{N}$ that contain it.

The elements of the set $\mathfrak{M}$ will be called the **vertices** of the complex, and the elements of the family $\mathfrak{N}$ the **simplices** of the complex. In this article, constructive locally finite simplicial complexes will be called **complexes**.

Denote by A^+ the standard two-letter extension of the alphabet A. Let us construct algorithms $\mathfrak{B}$ and $\mathfrak{C}$ such that, whatever the algorithm Φ in the alphabet A^+ transforming all elements of the set $\mathfrak{M}$ into constructive real numbers, and at least one of these elements into a number different from zero, the following conditions are satisfied:

1) the algorithm $\mathfrak{B}$ is applicable to the record** of the algorithm Φ, and

$$\mathfrak{B}(\{\Phi\}) \in \mathfrak{M} \;\&\; \Phi(\mathfrak{B}(\{\Phi\})) \neq 0;$$

2) the algorithm $\mathfrak{C}$ transforms the record of the algorithm Φ into the union of all finite sets that are members of the list $\mathfrak{A}(\mathfrak{B}(\{\Phi\}))$.

A **point of the complex** $(A, \mathfrak{M}, \mathfrak{N}, \mathfrak{A})$ will mean the record of any algorithm Φ in the alphabet A^{+} such that the following conditions are satisfied:

1) Φ transforms any vertex of the complex into a nonnegative constructive real number, and at least one vertex into a number different from zero;

2) whatever finite subset M of the set $\mathfrak{M}$ may be, if the result of applying Φ to each element of M is different from zero, then M belongs to the family $\mathfrak{N}$;

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* Here and below, by a constructive real number we mean a real duplex (a real FR-number) (see ([1])).

** The record of the algorithm Φ will be denoted by $\{\Phi\}$.

3) for any vertex P, if P does not belong to $\mathfrak{C}(\varphi)$, then $\Phi(P) = 0$;

4)
$$\sum_{P \in \mathfrak{C}(\varphi)} \Phi(P) = 1.$$

The **body of the complex** $(A, \mathfrak{M}, \mathfrak{N}, \mathfrak{A})$ will be called* the constructive metric space whose elements are the points of this complex, and whose metric function is an algorithm ρ such that, for any points φ and ψ,

$$\rho(\varphi \square \psi) = \left[\sum_{P \in M} (\Phi(P) - \Psi(P))^2 \right]^{1/2},$$

where M denotes the union of the finite sets $\mathfrak{C}(\varphi)$ and $\mathfrak{C}(\psi)$, and Φ and Ψ are algorithms in the alphabet A^{+} whose records are, respectively, the words φ and ψ.

Analogously to how this is done in classical mathematics (see ([2]), Ch. II), one introduces the notions of a linear mapping of one complex into another, of a simplicial mapping, of the open star of a complex and the closed star of a complex with a given vertex, of the direct product of complexes, of a linearly connected complex, of the n-dimensional skeleton of a complex, and of the n-fold barycentric subdivision of a complex. The term "subdivision" in this article will mean "barycentric subdivision."

By a **constructive operator** from a complex K into a complex L we shall mean a constructive operator from the body of the complex K into the body of the complex L.

Theorem 1. *For any constructive operator F from a complex K into a complex L, if F is applicable to at least one point of K, then one can construct a complex R, a linear mapping g of the complex R into K, and a simplicial mapping f of the complex R into L such that:*

1) *the mapping g is one-to-one;*

2) *for any point φ of K, if F is applicable to φ, then there is a potentially realizable point ψ of R such that $g(\psi) = \varphi$;*

3) *the mapping f is a simplicial approximation of the mapping $F \circ g$.**

Remark 1. For a uniformly continuous constructive operator F from a finite complex K into a complex L, defined on the entire body of the complex K, one may take as the complex R the n-fold subdivision of K for sufficiently large n.

Remark 2. One can construct a constructive operator from a finite complex K into a finite complex L, defined on the entire body of the complex K, for which the complex R occurring in the formulation of the theorem will necessarily be infinite. Examples of such an operator may be: a continuous but not uniformly continuous mapping of the segment $0\Delta 1$ into itself, constructed in the proof of Theorem 5.2 of ([4]), and a retraction of the square onto its boundary, constructed in the proof of Theorem 1 of ([5]).

In the proof of Theorem 1 the following lemma is used, whose proof differs only insignificantly from the proof of the main theorem of ([3]).

Lemma 1. *For any constructive operator F from a complex K into a complex L, one can construct algorithms σ, σ', and τ such that:*

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* The definitions of points and of the body of a complex formulated here correspond to the geometric realization of an abstract complex that is described in ([2]). This realization is called **standard** below.

** This theorem is a direct extension, to the case of a standardly realized complex, of G. S. Tseitin' s theorem on the approximation of constructive functions by pseudo-polygonal functions (Theorem 4 of ([3])).

1) whatever n may be, if σ is applicable to n, then σ' is applicable to n and $\sigma(n)$ is a vertex of the $\sigma'(n)$-fold subdivision of K;

2) whatever n may be, if τ is applicable to n, then $\tau(n)$ is a vertex of the complex L;

3) whatever n may be, if σ is applicable to n and F is applicable to a point φ belonging to the open star of the $\sigma'(n)$-fold subdivision of K with vertex $\sigma(n)$, then τ is applicable to n and the point $F(\varphi)$ belongs to the open star of the complex L with vertex $\tau(n)$;

4) whatever the point φ of K may be, if F is applicable to φ, then there is potentially realizable an n such that σ is applicable to n and the point φ belongs to the open star of the $\sigma'(n)$-fold subdivision with vertex $\sigma(n)$.

For the proof of Theorem 1, such regular* algorithms α, α', and β are constructed that:

1) whatever n may be, if α is applicable to n, then $\alpha(n)$ and $\beta(n)$ are natural numbers and σ is applicable to $\beta(n)$;
2) whatever n may be, $\alpha(n)$ is a vertex of the $\alpha'(n)$-fold subdivision of the complex K, and the closed star of this subdivision with vertex $\alpha(n)$ is contained in the open star of the $\sigma(\beta(n))$-fold subdivision of the same complex with vertex $\sigma(\beta(n))$;
3) whatever the point φ of K may be, if F is applicable to φ, then there is potentially realizable an n such that φ belongs to the open star of the $\alpha'(n)$-fold subdivision of K with vertex $\alpha(n)$;
4) whatever $n_1, n_2, \dots, n_k$ may be, if $\alpha'(n_1) \geq \alpha'(n_2) = \dots = \alpha'(n_k)$ and $\alpha(n_2), \dots, \alpha(n_k)$ belong to one simplex of the $\alpha'(n_2)$-fold subdivision of the complex K and are contained in the closed star of the $\alpha'(n_1)$-fold subdivision of K with vertex $\alpha(n_1)$, then $\tau(\beta(n_1)), \tau(\beta(n_2)), \dots, \tau(\beta(n_k))$ belong to one simplex of the complex L.

Denote by R_0 the complex whose set of vertices is the set of points of K enumerated by the algorithm α, and whose simplexes are the finite sets of the form $\{\alpha(n_1), \alpha(n_2), \dots, \alpha(n_k)\}$ such that: a) $\alpha'(n_1) \geq \alpha'(n_2) = \dots = \alpha'(n_k)$, and b) $\alpha(n_2), \dots, \alpha(n_k)$ belong to one simplex of the $\alpha'(n_2)$-fold subdivision of K and are contained in the closed star of the $\alpha'(n_1)$-fold subdivision of K with vertex $\alpha(n_1)$. Denote by g_0 the natural linear embedding of R_0 into the complex K, and by f_0 such a simplicial mapping of R_0 into L under which every vertex $\alpha(n)$ goes to the vertex $\tau(\beta(n))$ of the complex L. It is easy to see that f_0 is a simplicial approximation to the mapping $F \circ g$.

From Lemma 1 there also follows the following theorem.

Theorem 2. *Every constructive operator from one complex to another is continuous at every point to which it is applicable.*

We shall call a complex L an **algorithmic subcomplex** of a complex K if the set of its vertices is an algorithmically decidable subset of the set of vertices of the complex K, and the set of simplexes is an algorithmically decidable subset of the set of simplexes of the complex K. In this paper the term "subcomplex" everywhere means an algorithmic subcomplex.

Theorem 3. *Whatever the complex K, its subcomplex L, the linearly connected complex R, and the constructive operator F from L to R, applicable to all points of the complex L, one can construct an operator H from K to R, applicable to all points of the complex K and being an extension of the operator F to the complex K.*

In the proof of the theorem the following lemma will be used.

Lemma 2. *Whatever the complex I, if the dimension of I is greater than one, then one can construct a constructive operator from I to the boundary of I, leaving the boundary points fixed.*

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* An algorithm is called **regular** if, for every n, from the applicability of this algorithm to $n+1$ there follows its applicability to n.

This lemma follows from the potential realizability of a constructive retraction of the square onto its boundary (see (5)).

Let K_1, L_1 be the one-dimensional skeleta of the complexes K and L. Using an algorithm that recognizes whether a simplex belongs to the subcomplex L, and the linear connectedness of the complex R, we extend the operator F, considered only on L_1, to the complex K_1. Using the above-mentioned algorithm and Lemma 2, we extend the operator so obtained to the two-dimensional skeleton of the complex K. Continuing this process, and relying on the local finiteness of the complex K, we extend F to the whole complex K.

Remark. For the identity mapping of the boundary of a two-dimensional simplex into itself there can be no pseudouniformly continuous extension to the whole two-dimensional simplex (see (5)), although a continuous extension is realizable by virtue of the preceding theorem.

We shall say that operators f_1 and f_2 from a complex K to a complex L, applicable to all points of the complex K, are homotopic (uniformly homotopic) if there is a potentially realizable constructive (respectively, constructive uniformly continuous) operator F from the complex $K \times 0\Delta 1$ to L, applicable to all points of the complex $K \times 0\Delta 1$ and such that: a) at each point of the subcomplex $K \times 0$ the value F is equal to the value f_1; b) at each point of the subcomplex $K \times 1$ the value F is equal to the value f_2. Exactly as in classical mathematics, the relations of homotopy equivalence and uniform homotopy equivalence of two complexes are defined.

The following theorems follow from Theorem 3.

Theorem 4. *Any two constructive operators from a complex K to a linearly connected complex L, applicable to all points of the complex K, are homotopic.*

Theorem 5. *A complex is homotopy equivalent to a one-point space if and only if it is linearly connected.*

Remark. If in the formulations of Theorems 4 and 5 the words "homotopic" and "homotopy equivalent" are replaced by the words "uniformly homotopic" and "uniformly homotopy equivalent," then the resulting assertions will be false.

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Note: Figure translations are in progress. See original paper for figures.