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# Mathematics

Lai Duc Think

1963

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**Abstract**

**Full Text**

**Mathematics**

**Lai Duc Think**

## On the Number of Divisors in an Angle

*(Presented by Academician I. M. Vinogradov on 29 XII 1962)*

In <sup>(8)</sup> results were given concerning the sum

$$\sum_{\substack{N(\alpha) \leq x \\ \varphi_1 \leq \arg \alpha \leq \varphi_2}} \tau(\alpha),$$

where  $\alpha$  is an integer of an imaginary quadratic field, and  $\tau(\alpha)$  is the number of divisors of the number  $\alpha$ , with a remainder term equal to  $O(x^{2/3+\varepsilon})$ .

In the present paper we shall consider the same problem in real quadratic fields. Let  $\alpha$  be an integer of a real quadratic field  $k$  with discriminant  $d$ ;  $\tau(\alpha)$  is the number of divisors of  $\alpha$ . As in <sup>(8)</sup>, we note that

$$\sum_{|N(\alpha)|=n} \tau(\alpha) = O(n^\varepsilon)$$

for sufficiently large  $n$ . Put

$$\omega(\alpha) = \frac{1}{2 \ln \eta} \ln \left| \frac{\alpha}{\alpha'} \right|,$$

where  $\eta > 1$  is the fundamental unit of the field, and  $\alpha'$  is conjugate to  $\alpha$ . From each set of all numbers associated with  $\alpha$  we choose  $\alpha$  so that  $0 \leq \omega(\alpha) < 1$ . In this note results are given concerning the sum

$$\sum_{\substack{|N(\alpha)| \leq x \\ \omega(\alpha) \leq v < 1}} \tau(\alpha). \tag{1}$$

We shall consider two functions:

$$Z(s) = \sum_{\alpha} \frac{1}{|N(\alpha)|^s}, \quad Z(s, q) = \sum_{\alpha} \frac{\lambda^q(\alpha)}{|N(\alpha)|^s},$$

where  $\alpha$  runs through all integral non-associated numbers of the field  $k$ ,  $s = \sigma + it$ ,  $\sigma > 1$ , and  $\lambda(\alpha) = e^{2\pi i\omega(\alpha)}$ ,  $q \neq 0$  is a rational integer. For the remainder term of the sum (1), here too we shall obtain  $O(x^{2/3+\varepsilon})$ . Next we shall generalize the problem to  $\tau_\nu(\alpha)$ , namely, we shall compute the sums

$$\sum_{\substack{|N(\alpha)| \leq x \\ \omega(\alpha) \leq \nu < 1}} \tau_\nu(\alpha), \quad (2)$$

where  $\tau_\nu(\alpha)$  is the number of solutions of the equations  $\alpha_1, \alpha_2, \dots, \alpha_\nu = \alpha$ , when  $\alpha_1, \alpha_2, \dots, \alpha_\nu$  independently run through all non-associated divisors of  $\alpha$ , so that  $\tau_2(\alpha) = \tau(\alpha)$ . It should be noted that in imaginary fields we can also consider a sum analogous to the sum (2), and obtain the same estimates as here for the sum (2).

§ 1.

$$\sum_{|N(\alpha)| \leq x} \tau(\alpha).$$

With the aid of Theorem 101 from <sup>(1)</sup> it is easily proved that the fundamental ideal of every quadratic field is the principal ideal  $(\sqrt{d})$ . On this basis, as a special case of Theorem 154 from <sup>(3)</sup>, we obtain the following theorem:

**Theorem 1.** The function  $Z(s)$ , defined for  $\sigma > 1$ , can be analytically continued to the entire  $s$ -plane, satisfies the equation

$$\left(\frac{\sqrt{d}}{\pi}\right)^s \Gamma^2\left(\frac{s}{2}\right) Z(s) = \left(\frac{\sqrt{d}}{\pi}\right)^{1-s} \Gamma^2\left(\frac{1-s}{2}\right) Z(1-s) \quad (3)$$

and has a pole of the first order at the point  $s = 1$  with residue  $2 \ln \eta / \sqrt{d}$ .

**Theorem 2.** In the strip  $-\varepsilon \leq \sigma \leq 1 + \varepsilon$ , for  $|t| \geq 1$ , we have

$$Z(s) = O(|t|^{1-\sigma+\varepsilon}). \quad (4)$$

The proof of the theorem is based on the Phragmén–Lindelöf theorem <sup>(5)</sup>, on Theorem 1, and on estimates for the function  $\Gamma(s)$ .

**Theorem 3.**

$$\sum_{|N(\alpha)| \leq x} \tau(\alpha) = \frac{4 \ln^2 \eta}{d} x \ln x + \left(\frac{4c \ln \eta}{\sqrt{d}} - \frac{4 \ln^2 \eta}{d}\right) x + O(x^{3/5+\varepsilon}), \quad (5)$$

where

$$c = \lim_{s \rightarrow 1} \left\{ Z(s) - \frac{2 \ln \eta}{(s-1)\sqrt{d}} \right\}.$$

**Proof.** Applying the lemma on partial sums of a Dirichlet series <sup>(4)</sup>, Ch. III) under the assumption  $x = [x] + \frac{1}{2}$ , which entails no loss of generality, we obtain

$$\sum_{|N(\alpha)| \leq x} \tau(\alpha) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} Z^2(s) \frac{x^s}{s} dx + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

Moving the path of integration to the contour  $L\{1 + \varepsilon - iT, -\varepsilon - iT, -\varepsilon + iT, 1 + \varepsilon + iT\}$ , taking into account (3), (4), and Lemma 4, item 1, Ch. IV from <sup>(4)</sup>, and computing the residue  $\text{Res}_{s=1} Z^2(s) \frac{x^s}{s}$ , we obtain the required result by putting

$$T = x^{2/5}.$$

§ 2.  $\sum_{|N(\alpha)| \leq x} \tau(\alpha) \lambda_q(\alpha).$

**Theorem 4.** The function  $Z(s, q)$ , defined for  $\sigma > 1$ , can be analytically continued to the entire  $s$ -plane and satisfies the equation

$$\begin{aligned} & \left(\frac{\sqrt{d}}{\pi}\right)^s \Gamma\left(\frac{s}{2} + \frac{\pi i q}{2 \ln \eta}\right) \Gamma\left(\frac{s}{2} - \frac{\pi i q}{2 \ln \eta}\right) Z(s, q) = \\ & = \left(\frac{\sqrt{d}}{\pi}\right)^{1-s} \Gamma\left(\frac{1-s}{2} + \frac{\pi i q}{2 \ln \eta}\right) \Gamma\left(\frac{1-s}{2} - \frac{\pi i q}{2 \ln \eta}\right) Z(1-s, q). \end{aligned} \quad (6)$$

The theorem follows directly from equation (44) of <sup>(2)</sup>.

**Theorem 5.** In the strip  $-\varepsilon \leq \sigma \leq 1 + \varepsilon$ , for  $|t| \geq 1$ , we have

$$Z(s, q) = O\{|t| + |q|\}^{1-\sigma+\varepsilon}. \quad (7)$$

The proof is obtained on the basis of Theorem 4, estimates for the  $\Gamma$ -functions, and Theorem 2 from <sup>(6)</sup>.

**Theorem 6.**

$$\sum_{|N(\alpha)| \leq x} \tau(\alpha) \lambda^q(\alpha) = O\{(x^{1/3} + |q|)^{2+\varepsilon}\}. \quad (8)$$

The theorem is proved in the same way as Theorem 3, except that here the function  $Z(s, q)$  is analytic everywhere.

### § 3. The main theorem.

**Theorem 7.** In real quadratic fields we have

$$\sum_{\substack{|N(\alpha)| \leq x \\ \omega(\alpha) \leq v}} \tau(\alpha) = v \left\{ \frac{4 \ln^2 \eta}{d} x \ln x + \left( \frac{4c \ln \eta}{\sqrt{d}} - \frac{4 \ln^2 \eta}{d} \right) x \right\} + O(x^{2/3+\varepsilon}), \quad (9)$$

where  $0 < v < 1$ .

For the proof we shall use the following lemma, which is a modification of a lemma of I. M. Vinogradov from (7).

**Lemma.** Let  $v$  be a real number,  $0 < 2\Delta \leq v \leq 1 - 2\Delta$ ; then there exist two periodic functions  $\bar{f}(\omega)$  and  $\underline{f}(\omega)$  with period 1, having the following properties:

$$\bar{f}(\omega) = 1, \quad \text{if } 0 \leq \omega \leq v, \quad \underline{f}(\omega) = 1, \quad \text{if } \Delta \leq \omega \leq v - \Delta,$$

$$0 \leq \bar{f}(\omega) \leq 1, \quad \text{if } v \leq \omega \leq v + \Delta, \quad 1 - \Delta \leq \omega \leq 1,$$

$$0 \leq \underline{f}(\omega) \leq 1, \quad \text{if } v - \Delta \leq \omega \leq v, \quad 1 \leq \omega \leq 1 + \Delta,$$

$$\bar{f}(\omega) = 0, \quad \text{if } v + \Delta \leq \omega \leq 1 - \Delta; \quad \underline{f}(\omega) = 0, \quad \text{if } v \leq \omega \leq 1;$$

$\bar{f}(\omega)$  and  $\underline{f}(\omega)$  are expanded into Fourier series of the form

$$\bar{f}(\omega) = \sum_{q=-\infty}^{\infty} \bar{a}_q e^{2\pi i q \omega}, \quad \underline{f}(\omega) = \sum_{q=-\infty}^{\infty} \underline{a}_q e^{2\pi i q \omega},$$

where

$$\bar{a}_0 = v + \Delta, \quad \underline{a}_0 = v - \Delta,$$

$$\bar{a}_q \ll \frac{1}{|q|}, \quad \underline{a}_q \ll \frac{1}{|q|},$$

$$\bar{a}_q \ll \frac{1}{\Delta^r |q|^{r+1}}, \quad \underline{a}_q \ll \frac{1}{\Delta^r |q|^{r+1}},$$

$q \neq 0, \quad r \neq 0$  natural.

With the help of this lemma we have

$$\sum_{|N(\alpha)| \leq x} \tau(\alpha) \underline{f}[\omega(\alpha)] \leq \sum_{\substack{|N(\alpha)| \leq x \\ \omega(\alpha) \leq v}} \tau(\alpha) \leq \sum_{|N(\alpha)| \leq x} \tau(\alpha) \bar{f}[\omega(\alpha)].$$

Taking  $\Delta = x^{-1/3}$ ,  $r = 3$ , and summing the left- and right-hand sides of these inequalities with respect to  $q$ , we shall find formula (9).

#### § 4. Generalization for $\tau_\nu(\alpha)$ .

**Theorem 8.** In quadratic real fields we have

$$\sum_{\substack{|N(\alpha)| \leq x \\ \omega(\alpha) \leq v}} \tau_\nu(\alpha) = vxP_{\nu-1}(\ln x) + O(x^{\nu/(\nu+1)+\varepsilon}), \quad (10)$$

where  $P_\nu(y)$  is a polynomial of degree  $\nu$  in  $y$ .

The proof of formula (10) does not differ from the proof of formula (9). For the proof it is first necessary to compose analogues of formulas (5) and (8) for  $\tau_\nu(\alpha)$ , namely

$$\sum_{|N(\alpha)| \leq x} \tau_\nu(\alpha) = xP_{\nu-1}(\ln x) + O(x^{(2\nu-1)/(2\nu+1)+\varepsilon}),$$

$$\sum_{|N(\alpha)| \leq x} \tau_\nu(\alpha) \lambda^q(\alpha) = O\left\{ \left( x^{1/(\nu+1)} + |q| \right)^{\nu+\varepsilon} \right\}.$$

Next we apply the lemma with the condition  $\Delta = x^{-1/(\nu+1)}$  and  $r = \nu + 1$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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