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Abstract

Full Text

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R-Nets with Foliating Congruences of Axes

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In proving the existence in three-dimensional projective space of such nets that the congruences of their first and second axes are foliated respectively with the congruences of the first and second axes of the Laplace transforms of these nets, two different cases arise ([1]). With a suitable choice of frame, one of these classes of nets is described by the system:

$$\begin{gathered} \omega^3 = 0, \qquad \omega_2^1 = 0, \qquad d\ln\sigma = \omega_1^1 + \omega_3^3 - \omega_0^0 - \omega_2^2, \\ \omega_1^3 = a\omega^1, \qquad \omega_3^0 = 0, \qquad [\omega_2^0\omega^1] - c[\omega_3^1\omega^2] = 0, \quad (a) \\ \omega_2^3 = c\omega^2, \qquad \omega_3^2 = \sigma\omega^1, \qquad [\omega_2^0\omega^2] + a[\omega_3^1\omega^1] = 0. \quad (b) \\ \omega_1^2 = 0, \qquad \omega_1^0 = c\sigma\omega^2, \end{gathered} \tag{1}$$

In the present work the geometric properties of these nets are considered.

Theorem 1. *In order that a given net be a net described by system (1), it is necessary and sufficient that it be an R-net whose axes are the axes of at least one of its Laplace transforms.*

Proof. If M_0 is the given net, M_1 and M_{-1} are its Laplace transforms, N_2 is the Laplace transform of the net M_1, M_3 is the point of intersection of the ray $[M_1N_2]$ with the plane tangent to M_{-1}, and M_2 is the point of intersection of the ray $[M_0M_{-1}]$ with the plane tangent to N_2, then by virtue of system (1) $M_{-1} \equiv M_2$ and $M_3 \equiv N_2$. Thus, the axes of the net M_0 are the axes of the net M_1. From equations (1a) and (1b) it follows that

$$\{\omega_2^0 = aA_3\omega^1 + cA_4\omega_2, \quad \omega_2^1 = -A_4\omega^1 + A_3\omega^2\}. \tag{2}$$

From system (2) it follows that the asymptotic forms of the nets M_0, M_1, and M_2 are proportional, which proves the necessity of the assertion. If M_0 is an R-net whose axes coincide with the axes of the net M_1, then $M_3 \equiv N_2$ and $M_{-1} \equiv M_2$, and, choosing $[M_0M_1M_2M_3]$ as the frame, we obtain:

$$\{\omega^3 = 0,\ \omega_1^2 = 0,\ \omega_2^1 = 0,\ \omega_1^3 = a\omega^1,\ \omega_2^3 = c\omega^2\}. \tag{3}$$

Differentiating these equations and expanding by Cartan' s lemma, we obtain

$$\omega_1^0 = \beta\omega^2 + aA\omega^1, \qquad \omega_2^0 = m\omega^1 + cB\omega^2,$$
$$\omega_3^2 = \sigma\omega^1 - A\omega^2, \qquad \omega_3^1 = -B\omega^1 + l\omega^2,$$

whence the asymptotic forms of the nets M_0, M_1, and M_2 have the form:

$$\Phi_0 = a(\omega^1)^2 + c(\omega^2)^2, \quad \Phi_1 = a\sigma(\omega^1)^2 + \beta(\omega^2)^2, \quad \Phi_2 = m(\omega^1)^2 + cl(\omega^2)^2.$$

Since these forms are proportional, $\beta = c\sigma,\ m = al$. These equalities turn into identities the equations of foliation of the congruences $[M_0M_3]$ and $[M_1M_2]$, whence the theorem follows.

Consider the special case when in system (2) $A_4 = 0$. Joining equations (2), under the condition $A_4 = 0$, to system (1), and differentiating the resulting equations exteriorly, we shall have:

$$[\Delta \ln a; \omega^1] = 0, \qquad \Delta \ln a = d\ln a + \omega_0^0 - 2\omega_1^1 + \omega_3^3,$$
$$[\Delta \ln c; \omega^2] = 0, \qquad \Delta \ln c = d\ln c + \omega_0^0 - 2\omega_2^2 + \omega_3^3,$$
$$d\ln\sigma = \omega_1^1 + \omega_3^3 - \omega_0^0 - \omega_2^2,$$
$$d\ln A_3 = \omega_3^3 + \omega_2^2 - \omega_1^1 - \omega_0^0.$$

Adjoining the last two equations to system (1) and differentiating exteriorly, we shall have: $[\Delta \ln a; \omega^1] = 0$, $[\Delta \ln c; \omega^2] = 0$, whence $N = Q = S_1 = 2$. Consequently, the case $A_4 = 0$ exists with arbitrariness of two functions of one argument.

Theorem 2. *In order that the net M_0 belong to the case $A_4 = 0$, it is necessary and sufficient that the nets M_0, M_1, M_2, M_3 form a closed quadruple of R-nets.*

Proof. From equations (1) and (2), under the condition $A_4 = 0$, it follows that $dM_3 = \sigma\omega^1 M_2 + \omega_3^3 M_3$ (mod $\omega^2 = 0$). Similarly, $dM_2 = \omega_2^2 M_2 + c\omega^2 M_3$ (mod $\omega^1 = 0$). The converse is also true: every closed quadruple of R-nets satisfies the case $A_4 = 0$.

From the equation $[dF, F, M_0, M_3] = 0$ it follows that the foci of the congruence $[M_0M_3]$ have the form $F_1 = \sqrt{A_3\sigma}\, M_0 + M_3$, $F_2 = -\sqrt{A_3\sigma}\, M_0 + M_3$, whence $(F_1F_2; M_0M_3) = -1$. In this case the equation of the return edge F_1 has the form $\sqrt{\sigma}\omega^1 + \sqrt{A_3}\omega^2 = 0$, and the return edge F_2 has the form $\sqrt{\sigma}\omega^1 - \sqrt{A_3}\omega^2 = 0$. The requirement that they be conjugate with respect to the form Φ_0 leads to the condition $aA_3 = c\sigma$, which is equivalent to the requirement

$$[\omega_1^0\omega^1] + [\omega_2^0\omega^2] = 0. \tag{*}$$

Let $\omega^1 = \alpha_1 du$, $\omega^2 = \alpha_2 dv$, $\omega_0^0 = \alpha_3 du + \alpha_4 dv$, $\omega_1^1 = \alpha_5 du + \alpha_6 dv$. Then from $dM_0 = \omega_0^0 M_0 + \omega^1 M_1 + \omega^2 M_2$ and $dM_1 = \omega_1^0 M_0 + \omega_1^1 M_1 + \omega_1^3 M_3$ we have $M_{0u} = \alpha_3 M_0 + \alpha_1 M_1$ and $M_{1v} = \beta\alpha_2 M_0 + \alpha_6 M_1$, whence we have $M_{0uv} = \{\alpha_6 + (\ln\alpha_1)_v\}M_{0u} + \alpha_3 M_{0v} + \{\alpha_{3v} + \beta\alpha_1\alpha_2 - \alpha_3(\ln\alpha_1)_v - \alpha_3\alpha_5\}M_0$.

The Darboux invariants are written as ([2]):

$$h = \alpha_{3v} + \beta\alpha_1\alpha_2 - \alpha_{6u} - (\ln\alpha_1)_{uv}, \qquad j = \beta\alpha_1\alpha_2.$$

Consequently, $h-j = \alpha_{3v}-\alpha_{6u}-(\ln\alpha_1)_{uv}$. From the equality $D\omega^1 = [\omega_0^0-\omega_1^1\,\omega^1]$ we obtain $(\ln\alpha_1)_v = \alpha_4 - \alpha_6$, consequently, $(\ln\alpha_1)_{uv} = \alpha_{4u} - \alpha_{6u}$. Conditions $(*)$ give $D\omega_0^0 = 0$, i.e. $\alpha_{3v} = \alpha_{4u}$. Consequently, $(\ln\alpha_1)_{uv} = \alpha_{3v} - \alpha_{6u}$ and $h = j$. But if M_0 is a net with equal invariants, then M_3 is also a net with equal invariants ([2]). From equation $(*)$ it follows that the net M_1, and hence also M_2, is also a net with equal invariants. Analogous assertions are true also for the congruence $[M_1M_2]$. The foci of the ray $[M_1M_2]$ have the form: $F_0 = \sqrt{A_3}M_1 + \sqrt{\sigma}M_2$; $F_3 = \sqrt{A_3}M_1 - \sqrt{\sigma}M_2$; the equation of the return edge F_0 is $a\sqrt{A_3}\omega^1 + c\sqrt{\sigma}\omega^2 = 0$, the equation of the return edge F_3 is $a\sqrt{A_3}\omega^1 - c\sqrt{\sigma}\omega^2 = 0$. If M_0 is a net with equal invariants, then the return edge F_0 corresponds to the return edge F_1, and the return edge F_2 corresponds to the return edge F_3. Consequently, in this case the net M_0 is harmonic to the congruence $[M_1M_2]$. From what has been said it follows

Theorem 3. *In the case $A_4 = 0$, the foci of the congruence $[M_0M_3]$ harmonically separate the points M_0 and M_3. If M_0 is a net with equal Darboux invariants, then the net M_0 is conjugate to the congruence $[M_0M_3]$ and harmonic to the congruence $[M_1M_2]$. Analogous assertions are true for the net M_1.*

The equations of the net M_0 with equal invariants in the case $A_4 = 0$ have the form:

$$\begin{array}{llllll} \omega^3 = 0, & \text{(a)} & \omega_1^2 = 0, & (\ ) & \omega_3^2 = \sigma\omega^1, & (\ ) \\ \omega_1^3 = a\omega^1, & (\ ) & \omega_2^1 = 0, & (\ ) & \omega_1^0 = c\sigma\omega^2, & (\ ) \\ \omega_2^3 = c\omega^1, & (\ ) & \omega_3^0 = 0, & (\ ) & \omega_2^0 = c\sigma\omega^1, & (\ ) \\ a\omega_3^1 = c\sigma\omega^2, & (\ ) & d\ln\sigma = \omega_0^1 + \omega_3^3 - \omega_0^0 - \omega_2^2. & (\ ) & & \end{array} \tag{4}$$

Differentiating (4), (4), (4), and (4), we shall have

$$d\ln a = 2\omega_1^1 - \omega_0^0 - \omega_3^3, \qquad d\ln c = 2\omega_2^2 - \omega_0^0 - \omega_3^3.$$

These equations must be adjoined to system (4), and then the new system will prove to be completely integrable.

From the expressions for F_1, F_2, F_3, F_0 we obtain

$$[dF_1, F_1, F_0] = [dF_0, F_1, F_0] = 0.$$

Consequently, the line $[F_0F_1]$ is fixed; analogously, the line $[F_2F_3]$ is fixed. Thus the axes of the net M_0 belong to one linear congruence, whose directrices are

F_0F_1 and F_2F_3. Under displacement along the line $\sqrt{\sigma}\omega^1+\sqrt{A_3}\omega^2=0$, all first axes of the net M_0 pass through F_1, while under displacement along the line $\sqrt{\sigma}\omega^1-\sqrt{A_3}\omega^2=0$, all first axes of the net M_0 pass through F_2. Similarly, under displacement along the line $a\sqrt{A_3}\omega^1+c\sqrt{\sigma}\omega^2=0$, all second axes of the net pass through F_0, while under displacement along the line $a\sqrt{A_3}\omega^1-c\sqrt{\sigma}\omega^2=0$, all second axes of the net M_0 pass through the point F_3. The first axes pass through $F_1(F_2)$ and the second axes pass through $F_0(F_3)$ simultaneously if and only if the net M_0 is a net with equal invariants.

Differentiating equations (2) externally, we obtain

$$\Delta A_4 = B_1\omega^1 + B_2\omega^2, \qquad \Delta A_4 = dA_4 - A_4(\omega_3^3 - \omega_0^0),$$

$$a\Delta A_3 = -aB_2\omega^1 + cB_1\omega^2, \qquad \Delta A_3 = dA_3 + A_3(\omega_0^0 + \omega_1^1 - \omega_2^2 - \omega_3^3).$$

Putting $B_1 = B_2 = 0$, we shall have:

$$d\ln A_3 = \omega_2^2 + \omega_3^3 - \omega_0^0 - \omega_1^1, \qquad d\ln A_4 = \omega_3^3 - \omega_0^0.$$

Adjoining these equations to systems (1) and (2), we obtain a system in involution, and the arbitrariness of the solution depends on two functions of one argument.

Theorem 4. If $B_1 = B_2 = 0$, then all Laplace transforms of the net with even indices lie on the ray $[M_0M_3]$, and the transforms with odd indices lie on the ray $[M_1M_2]$.

Proof. Denote $M_0 = L_0,\ M_1 = L_1,\ M_2 = L_{-1},\ M_3 = L_2$. Then

$$dL_{-1} \pmod{\omega^1 = 0} = A_4M_0 + cM_3 = L_{-2},$$

$$dL_{-2} \pmod{\omega^2 = 0} = \omega_3^2L_{-1} + \omega_3^3L_{-2},$$

whence it follows that L_{-2} is the focus of the ray $[L_{-1}L_{-2}]$. If we denote $N_0 = L_{-1},\ N_1 = L_0,\ N_2 = L_2,\ N_3 = L_1$, then the forms of infinitesimal displacements of the frame $[N_0N_1N_2N_3]$, θ_i^k, satisfy equations analogous to systems (1), (2), and (6); here $\bar{a} = 1/aA_3,\ \bar{c} = A_3/c,\ \bar{\sigma} = 1/A_3,\ \bar{A}_3 = \sigma,\ \bar{A}_4 = A_4/A_3$. Thus the assertion of the theorem is true for the next net, since it is true for the preceding one. The forms π_i^k of the infinitesimal displacements of the frame $[L_1L_2L_0L_3]$ satisfy a system of equations analogous to equations (1), (2), and (6). In this case $\bar{a} = 1/a,\ \bar{c} = 1/c\sigma^2,\ \bar{\sigma} = \sigma A_3,\ \bar{A}_3 = 1,\ \bar{A}_4 = A_4$, whence the validity of the theorem follows.

The equation of the foci of the ray $[M_0M_3]$ gives $F_1 = \lambda_1 M_0 + M_3$, $F_2 = \lambda_2 M_0 + M_3$, where λ_1 and λ_2 are the roots of the equation $\lambda^2 - A_4\lambda - A_3\sigma = 0$. The equations of the developable surfaces are $\sigma\omega^1 + \lambda_1\omega^2 = 0$ and $\sigma\omega^1 + \lambda_2\omega^2 = 0$. Similarly, the foci F_0 and F_3 of the ray $[M_1M_2]$ are written in the form $F_{0(3)} = \mu_{1(2)}M_1 + M_2$, where μ_1 and μ_2 are the roots of the equation $\sigma\mu^2 + A_4\mu - A_3 = 0$. The equation of the return edge F_0 is $a\mu_1\omega^1 + c\omega^2 = 0$, the equation of the return edge F_3 is $a\mu_2\omega^1 + c\omega^2 = 0$. If we require that the lines $\sigma\omega^1 + \lambda_1\omega^2 = 0$ and $\sigma\omega^2 + \lambda_2\omega^2 = 0$ be conjugate on the surface M_0, then $aA_3 = c\sigma$, which means equality of the invariants of the net M_0. In this case the lines $\sigma\omega^1 + \lambda_1\omega^2 = 0$ and $a\mu_1\omega^1 + c\omega^2 = 0$ coincide, whence it follows

Theorem 5. The net M_0 is conjugate to the congruence $[M_0M_3]$ and to the harmonic congruence $[M_1M_2]$ if and only if M_0 is a net with equal invariants.

The nets described in the condition of Theorem 5 satisfy a completely integrable system.

The expression $A_4^2 + 4A_3\sigma$ is a relative invariant of the net M_0, since

$$d(A_4^2 + 4A_3\sigma) = 2(\omega_3^3 - \omega_0^0)(A_4^2 + 4A_3\sigma).$$

Its vanishing at 0 means that the congruences of axes of the nets M_0 and M_1 are parabolic. From the expressions for F_1, F_2, F_3, F_0 we have

$$dF_1 = \omega_3^3 F_1 + \theta F_0,$$

where θ is a form proportional to the form $\sigma\omega^1 + \lambda\omega^2$, and

$$dF_0 = \omega_2^2 F_0 + \nu F_1,$$

where ν is a form proportional to the form $a\mu_1\omega^1 + c\omega^2$, whence it follows

Theorem 6. *The axes of the net M_0 belong to one and the same linear congruence, whose directrices are F_1F_0 and F_2F_3. The developable surfaces of the congruence of the first and second axes degenerate into cones with vertices at F_0, F_1, F_2, F_3, whose equations are:*

$$a\mu_1\omega^1 + c\omega^2 = 0, \qquad \sigma\omega^1 + \lambda_1\omega^2 = 0, \qquad a\mu_2\omega^1 + c\omega^2 = 0, \qquad \sigma\omega^1 + \lambda_1\omega^2 = 0.$$

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Note: Figure translations are in progress. See original paper for figures.