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Abstract

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SPECTRAL PROPERTIES OF THE SCHRÖDINGER OPERATOR IN DOMAINS WITH INFINITE BOUNDARY

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In the present communication we set forth some of the results obtained in the study of the spectrum of the Schrödinger operator in domains with infinite boundary. We shall consider the planar case.

1°. Let

$$[\Delta + k^2 - p(x)]u = 0 \quad \text{in } D, \qquad k^2 > 0; \tag{1}$$

$$u|_\Gamma = 0; \tag{2}$$

$$\limsup_{r\to\infty} rq(r) = 2k\mu, \qquad \text{where } q(r) = \sup_{|x|=r,\ x\subset D} |p(x)|; \tag{3}$$

$$M^2(r) \equiv \int_{|x|=r,\ x\subset D} |u|^2\, d\Sigma, \qquad N^2(r) \equiv \int_{|x|=r,\ x\subset D} \left|\frac{\partial u}{\partial r}\right|^2 d\Sigma. \tag{4}$$

With respect to the domain D and its boundary Γ we shall make the following assumptions. We shall regard the boundary Γ as smooth. Suppose, further, that outside some circle of arbitrarily large, but fixed, radius R_0, the radius vector, intersecting the boundary Γ, remains inside the domain D. In other words, we require that for $r > R_0$, $\delta\Sigma_{1r}/\delta r > 0$, where $\delta\Sigma_{1r}$ is the spherical image of that part of the boundary Γ which is situated in the region $r \le |x| \le r + \delta r$. The coefficient $p(x)$ will be assumed to be a bounded complex-valued function.

Under the assumptions made, the following theorems are valid:

Theorem 1. Let $u \neq 0$ satisfy equation (1) and the boundary condition (2). If $\mu < 1$, then

$$\lim_{r\to\infty} r^{2\mu+\varepsilon}[k^2M^2(r) + N^2(r)] = \infty; \tag{5}$$

$$\lim_{R\to\infty} R^{2\mu+\varepsilon} \int_R^{R+b} M^2(r)\,dr = \infty \tag{6}$$

for any fixed $\varepsilon > 0,\ b > 0$. Every solution of problem (1)–(2) belonging to $L_2(D)$, for $\mu < \frac{1}{2}$, is identically zero;

Theorem 2. Let $\mu < 1,\ u \neq 0$

$$\int_{R_0}^{\infty} q(r)\,dr < \infty. \tag{7}$$

Then

$$\liminf_{r\to\infty} [k^2 M^2(r) + N^2(r)] > 0; \tag{8}$$

$$\liminf_{R\to\infty} \int_R^{R+b} M^2(r)\,dr > 0 \tag{9}$$

for any fixed b. Every solution of problem (1)–(2) for which $M(r) \to 0$ or $u(x) \subset L_2(D)$ is identically zero.

Remark 1. Theorems 1 and 2 are transferred without changes in the formulations and proofs to the case of an n-dimensional domain. Theorems 1 and 2 generalize

extend the results of paper ([1]) to the case of a domain with an infinite boundary and are proved by the method given in ([1]).

2°. Suppose that the domain D satisfies the conditions of item 1° and its boundary Γ approaches the boundary Γ_0 of the angle D_0 in such a way that

$$\rho(s, \Gamma_0) < \frac{c}{1 + |s|^{1+\alpha}}, \quad \alpha > 0, \tag{10}$$

where $\rho(s, \Gamma_0)$ is the distance from the point $s \subset \Gamma$ to Γ_0, and by c here and below different constants are denoted. In the indicated domain let us pose the problem of constructing the resolvent kernel of the Schrödinger operator:

$$[\Delta + (k + i\varepsilon)^2 - p(x)] G(x, y; k + i\varepsilon) = \delta(x - y) \text{ in } D \quad k^2 > 0,\ \varepsilon > 0, \tag{11}$$

$$G(x, y; k + i\varepsilon)\big|_{x\subset\Gamma} = 0; \tag{12}$$

$$G(x, y; k + i\varepsilon) \subset L_2(D) \tag{13}$$

under the assumption

$$|p(x)| < \frac{c}{1 + |x|^{2+\beta}}, \quad \beta > 0. \tag{14}$$

Theorem 3. *Under the assumptions made concerning the domain D, problem (11)–(13) has a unique solution, and this solution has the following properties:*

a) *as $\varepsilon \to 0$, $G(x, y; k + i\varepsilon)$ tends uniformly (when x, y, k vary in finite domains) to its limiting values $G(x, y, k)$;*

b) *the function $G(x, y; k) = G(y, x; k)$ satisfies equation (11) for $\varepsilon = 0$, the boundary condition (12) $(\varepsilon = 0)$, and the "radiation condition" ([2])*

$$\int_{\Sigma_R} \left| \frac{\partial G}{\partial n} - ikG \right|^2 d\Sigma \xrightarrow[R \to \infty]{} 0,$$

c)

$$|G(x, y; k)| < |C \ln |x - y||, \quad |\nabla G(x, y; k)| < \frac{c}{|x - y|} \quad \text{as } |x - y| \to 0;$$

$$|G(x, y; k)| < \frac{C}{|x - y|^{1/2}}, \quad |\nabla G(x, y; k)| < \frac{C}{|x - y|^{1/2}} \quad \text{as } |x - y| \to \infty.$$

Theorem 4. *The function $G(x, y; k)$ constructed in Theorem 3 admits the expansion*

$$G(x, y; k) = \frac{1}{4i} \sqrt{\frac{2}{\pi k |y|}} \, e^{i(k|y| - \pi/4)} u(x, \omega, k)(1 + o(1)) \tag{15}$$

$$\substack{|y| \to \infty \\ \arg y = \omega}$$

(the variable ω varies inside the angle D_0). The functions $u(x, \omega, k)$ satisfy equation (11) and condition (12) (for $\varepsilon = 0$).

Remark 2. From Theorems 3 and 4 it follows that the positive spectrum of the Schrödinger operator in the domain under consideration is absolutely continuous. The functions $u(x, \omega, k)$ are analogous to the "plane waves" for the domain D. It is natural to call them solutions of the scattering problem on the potential $p(x)$ in the domain D with boundary condition (12), ([4]).

The spectral function of the Schrödinger operator is weakly differentiable with respect to λ, and $dE_\lambda/d\lambda$ is an integral operator with kernel $\frac{1}{\pi}\operatorname{Im} G(x,y,\sqrt{\lambda})$.

Theorem 5. *The following mutually inverse formulas hold*

$$f(x)=\frac{1}{2\pi}\int_{D_0}\hat{f}(k)u(x,k)\,dk+\sum_{p=1}^{n}c_p\varphi_p(x).$$

$$\hat{f}(k)=\frac{1}{2\pi}\int_D f(x)\overline{u(x,k)}\,dx,\quad c_p=\int_D f(x)\varphi_p(x)\,dx. \tag{16}$$

The integrals in (16) are understood as limits in the mean (an analogue of Fourier integrals); $\varphi_p(x)$ are eigenfunctions of the negative spectrum of problem (1)–(2), which, under the assumptions made, is discrete and finite ([3]); $\mathbf{k}=k\omega$, $\tilde{f}(\mathbf{k})\in L_2(D_0)$, $f(x)\in L_2(D)$.

3°. In this subsection we shall establish a connection between the spectral properties of the operator and the behavior of solutions of the nonstationary problem. Suppose that the function $G(x,y;k)$, constructed in Theorem 3, admits the estimate:

$$\left|\int_D G(x,y;k)f(y)\,dy\right|<\frac{C}{1+|k|^a},\qquad a>0,\ c=c_f=\text{const} \tag{17}$$

for smooth, sufficiently rapidly decreasing $f(y)$. Estimate (17) was obtained for the case $D=E_3$ in ([6]). From the results of ([7]) one can obtain an analogous estimate when D is the exterior of a plane bounded convex domain or an angular domain. Under the assumption that (17) holds, the following assertions are true.

Consider the nonstationary problem:

$$u_{tt}+Lu=f(x)e^{i\omega t}\text{ in } D,\qquad \omega>0; \tag{18}$$

$$u\big|_{t=0}=0,\qquad u_t\big|_{t=0}=\varphi(x); \tag{19}$$

$$u\big|_\Gamma=0; \tag{20}$$

$$Lu\equiv-\Delta u+p(x)u, \tag{21}$$

where

$$\{|f(x)|,\ |\varphi(x)|\}<\frac{C}{1+|x|^{2+b}},\qquad b>0. \tag{22}$$

Assume also that

$$(Lu, u) > 0. \tag{23}$$

Under the assumptions made, the following theorems hold:

Theorem 6. Let $f(x) \equiv 0$. Then

$$\frac{1}{T}\int_0^T u(x,t)\,dt = o\left(\frac{1}{T^{1-\sigma}}\right),$$

where $\sigma > 0$ is arbitrarily small.

Theorem 7. Let $f(x) \not\equiv 0$. Then

a) there exists

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T u(x,t)e^{-i\omega t}\,dt = v(x),$$

$$\frac{1}{T}\int_0^T \left[e^{-i\omega t}u(x,t) - v(x)\right]dt = o\left(\frac{1}{T^{1-\sigma}}\right)$$

($\sigma > 0$ arbitrarily small);

b) the function $v(x)$ is a solution of the stationary problem:

$$Lv - \omega^2 v = f(x); \tag{24}$$

$$v\big|_\Gamma = 0; \qquad \int_{\Sigma_R}\left|\frac{\partial v}{\partial n} - ikv\right|^2 d\Sigma \xrightarrow[R\to\infty]{} 0. \tag{25}$$

Remark 3. By the method given in ([5]), one can prove that

$$\lim_{t\to\infty} e^{-i\omega t}u(x,t) = v(x), \tag{26}$$

where $v(x)$ is the solution of problem (24)–(25).

If we retain the assumptions of item 3°, with the exception of (23), and set $f(x) \equiv 0$, then the following theorems hold:

Theorem 8. *The operator L has no negative spectrum if and only if the solution of problem (18)–(20) admits the estimate*

$$\left|\int_0^t u(x,\tau)\,d\tau\right| \le O(e^{\varepsilon t}), \tag{27}$$

where the point x is fixed in D, and $\varepsilon > 0$ is arbitrarily small.

Theorem 9. *The operator L has no positive discrete spectrum when, as $t \to \infty$,*

$$\int_0^t u(x,\tau) e^{-i\lambda\tau}\,d\tau = o(t) \quad \text{for all } \lambda \ge 0; \tag{28}$$

the point $\lambda \ge 0$ does not belong to the discrete spectrum of the operator when (28) is fulfilled for $\lambda = \lambda_0$.

Remark 4. In Theorems 8 and 9 we assume that the estimates (27), (28) hold under the condition that in problem (18)–(20), as $\varphi(x)$, we take the set of smooth finite functions.

Here the constants in (27), (28) may depend on $\varphi(x)$.

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Note: Figure translations are in progress. See original paper for figures.