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Abstract

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A MIXED PROBLEM FOR A HYPERBOLIC EQUATION WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVES

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The aim of our note is to construct an asymptotic expansion, with respect to the small parameter, of the solution of the following mixed problem for a second-order hyperbolic equation in the cylinder $Q = \Omega \times [0 \leq x_0 \leq T]$:

$$\mathcal{L}_\varepsilon u \equiv \varepsilon \mathcal{L}_1 u + \mathcal{L}_2 u = f; \tag{1}$$

$$u(x_0, \mathbf{x})\big|_{x_0=0} = 0, \qquad \frac{\partial u(x_0, \mathbf{x})}{\partial x_0}\bigg|_{x_0=0} = 0; \tag{2}$$

$$u\big|_F = 0, \tag{3}$$

where $\varepsilon > 0$ is a small parameter; Ω is an n-dimensional domain with boundary S; $F = S \times [0 \leq x_0 \leq T]$ is the lateral surface of the cylinder Q, $\mathbf{x} = (x_1, \dots, x_n)$;

$$\mathcal{L}_1 u \equiv \sum_{i,j=0}^{n} a_{ij}(x_0, \mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}; \qquad \mathcal{L}_2 u \equiv \sum_{i=0}^{n} a_i(x_0, \mathbf{x}) \frac{\partial u}{\partial x_i} + b(x_0, \mathbf{x}) u,$$

$a_{00} = 1;\ a_{ij} = a_{ji}$, and the hyperbolicity condition is satisfied.

For $\varepsilon = 0$, equation (1) degenerates into the equation $\mathcal{L}_2 w = f$. Having specified the positive direction of the characteristics $l(x_0, \mathbf{x})$ of the equation $\mathcal{L}_2 w = f$, we split F into two parts F_1 and F_2. Let F_1 be the part into which the characteristics enter, and F_2 the part from which they leave.

We shall seek solutions of (1), (2), and (3) in the form

$$u = \sum_{i=0}^{m} \varepsilon^i w_i + \sum_{i=0}^{m} \varepsilon^{i+1} (v_i^1 + v_i^2) + \varepsilon^{m+1} Z_m, \tag{4}$$

where the functions w_i are determined by the first iterative process, and the functions v_i^1 and v_i^2 by the second iterative processes; Z_m is the remainder term.

Before defining the functions occurring in (4), let us give the second splitting of the operator $\mathcal{L}_\varepsilon$ near the boundary $x_0 = 0$ and F_2.

In the first case we make the change of variables $x_0 = \varepsilon\tau$ and expand the coefficients of $\mathcal{L}_\varepsilon$ in powers of $\varepsilon\tau$. Substituting this expansion into the expression $\mathcal{L}_\varepsilon$ and grouping terms with equal powers of ε, we obtain

$$\mathcal{L}_\varepsilon \equiv \varepsilon^{-1}\left(M_0^1 + \sum_{s=1}^{m+1} \varepsilon^s M_s^1\right), \qquad \text{where} \quad M_0^1 \equiv \frac{\partial^2}{\partial\tau^2} + a_0^0(\mathbf{x})\frac{\partial}{\partial\tau}$$

($a_0^0(\mathbf{x})$ is the first term in the expansion of the coefficient $a_0(x_0, \mathbf{x})$ in powers of $\varepsilon\tau$).

In order to write the analogous splitting of the operator $\mathcal{L}_\varepsilon$ near the boundary F_2, in a ρ_0-neighborhood of F_2, we introduce local coordinates $(x_0, \mathbf{y}, \rho)$ as follows. Let M_1 be an arbitrary point of the surface F_2, and M_2 an interior point of Q. For ρ we take the length of such a vector $\overrightarrow{M_1M_2}$ that

$$\left|\cos(\vec{n}, \overrightarrow{M_1M_2})\right| \geq \delta > 0,$$

where δ is a fixed number for all points M_1; $\vec{n}$ is the inward normal. By $\mathbf{y} = (y_1, \dots, y_{n-1})$ we denote coordinates on the surface S_2, which is the projection of F_2 onto $x_0 = 0$.

In the new coordinates the operator $\mathcal{L}_\varepsilon$ has the form

$$\mathcal{L}_\varepsilon \equiv \varepsilon A(x_0, \mathbf{y}, \rho)\frac{\partial^2}{\partial\rho^2} + B(x_0, \mathbf{y}, \rho)\frac{\partial}{\partial\rho} + h\left(\varepsilon, x_0, \mathbf{y}, \rho, \frac{\partial}{\partial\rho}, \dots\right),$$

where $h(\varepsilon, x_0, \mathbf{y}, \rho, \partial/\partial\rho, \dots)$ is a known function.

In the last expression $\mathcal{L}_\varepsilon$ we make the change of variables $\rho = \varepsilon\eta$ and expand the coefficients in powers of $\varepsilon\eta$. Analogously to the preceding, we obtain:

$$\mathcal{L}_\varepsilon \equiv \varepsilon^{-1}\left(M_0^2 + \sum_{i=1}^{m+1} \varepsilon^i M_i^2\right), \qquad \text{where } M_0^2 \equiv A^0\frac{\partial^2}{\partial\eta^2} + B^0\frac{\partial}{\partial\eta}$$

(A^0 and B^0 are the first terms of the expansions of $A(x_0, y, \rho)$ and $B(x_0, y, \rho)$ in powers of ε, η, respectively, and $A^0B^0 > 0$).

In the first iterative process the approximate solution of equation (1) is sought in the form

$$u = w_0 + \sum_{i=1}^{m} \varepsilon^i w_i.$$

Substituting this expression for $u(x_0, \mathbf{x})$ into equation (1) and comparing terms with equal powers of ε, we obtain the following system of equations:

$$\mathcal{L}_2 w_0 = f; \tag{5}$$

$$\mathcal{L}_2 w_i = -\mathcal{L}_1 w_{i-1}, \qquad i = 1, 2, \dots, m. \tag{6}$$

In the second iterative process, near the boundary $x_0 = 0$, the approximate solution of the homogeneous equation $\mathcal{L}_\varepsilon v = 0$ is sought in the form

$$v = \varepsilon v_0' + \sum_{i=1}^{m} \varepsilon^{i+1} v_i^1.$$

Substituting the expression for v into the equation and comparing terms with equal powers of ε, we obtain

$$M_0^1 v_0^1 = 0; \tag{7}$$

$$M_0^1 v_i^1 = -\sum_{j=1}^{i} M_j^1 v_{i-j}^1, \qquad i = 1, \dots, m. \tag{8}$$

In an analogous manner, near the boundary F_2 we obtain the system

$$M_0^2 v_0^2 = 0; \tag{9}$$

$$M_0^2 v_i^2 = -\sum_{j=1}^{i} M_j^2 v_{i-j}^2, \qquad i = 1, \dots, m. \tag{10}$$

Obviously, the results of the iterative processes at each stage are connected with one another by initial and boundary conditions. To reveal this connection, we substitute the expression $u(x_0, \mathbf{x})$ from (4) into (2) and (3) and compare terms with equal powers of ε. As a result we obtain

$$w_0\big|_{x_0=0} = 0, \qquad w_i\big|_{x_0=0} = -v_{i-1}^1\big|_{\tau=0} - v_{i-1}^2\big|_{x_0=0}, \qquad i = 1, 2, \dots, m; \tag{11}$$

$$Z_m\big|_{x_0=0} = \varphi_1, \tag{12}$$

where φ_1 is a known function,

$$\left.\frac{\partial w_0}{\partial x_0}\right|_{x_0=0} = -\left.\frac{\partial v_0^1}{\partial \tau}\right|_{\tau=0}; \tag{13}$$

$$\left.\frac{\partial v_i^1}{\partial \tau}\right|_{\tau=0} = -\left(\frac{\partial w_i}{\partial x_0} + \frac{\partial v_{i-1}^2}{\partial x_0}\right)\Bigg|_{x_0=0}, \qquad i = 1, \dots, m; \tag{14}$$

$$\left.\frac{\partial Z_m}{\partial x_0}\right|_{x_0=0} = \varphi_2, \tag{15}$$

where φ_2 is a known function, and, finally,

$$w_i = -(v_{i-1}^1 + v_{i-1}^2)\big|_{F_1}, \qquad i = 0, 1, \dots, m; \tag{16}$$

$$v_i^2\big|_{\eta=0} = -(w_i + v_{i-1}^1)\big|_{F_2}, \qquad i = 0, 1, \dots, m; \tag{17}$$

$$Z_m\big|_F = \varphi_3, \tag{18}$$

where φ_3 is a known function. Here it is assumed that if $r > 0$, then $v_{-r}^i = 0$ $(i = 1, 2)$.

Now define the functions w_i, v_i^1, and v_i^2 $(i = 0, 1, \dots, m)$. The function w_0 is the solution of the problem

$$\mathcal{L}_2 w_0 = f; \qquad w_0\big|_{x_0=0} = 0, \qquad w_0\big|_{F_1} = 0.$$

Let f, together with its derivatives up to the order needed for us, vanish along the characteristic $l((0, 0))$. Then the problem for w_0 has a unique smooth solution.

As soon as w_0 becomes known, v_0^1 is determined from the following problem:

$$M_0^1 v_0^1 \equiv \frac{\partial^2 v_0^1}{\partial \tau^2} + a_0^0(x)\frac{\partial v_0^1}{\partial \tau} = 0, \qquad \left.\frac{\partial v_0^1}{\partial \tau}\right|_{\tau=0} = -\left.\frac{\partial w_0}{\partial x_0}\right|_{x_0=0}.$$

Obviously,

$$v_0^1 = c_0^1(x)e^{-a_0^0(x)\tau} = c_0^1(x)e^{-a_0^0(x)\frac{x_0}{\varepsilon}},$$

where

$$c_0^1(x) = \frac{1}{a_0^0(x)} \frac{\partial w_0}{\partial x_0}\bigg|_{x_0=0}.$$

Let $a_0^0(x) > 0$. Under this condition v_0^1 is a boundary-layer type function near the boundary $x_0 = 0$.

Now define the function v_0^2. It is the solution of the following problem:

$$M_0^2 v_0^2 \equiv A^0 \frac{\partial^2 v_0^2}{\partial \eta^2} + B^0 \frac{\partial v_0^2}{\partial \eta} = 0, \qquad v_0^2\big|_{F_2} = -w_0\big|_{F_2}.$$

Hence

$$v_0^2 = c_0^2(x_0, y)e^{-\frac{B^0}{A^0}\eta} = c_0^2(x_0, y)e^{-\frac{B^0}{A^0}\frac{\rho}{\varepsilon}},$$

where $c_0^2(x_0, y) = -w_0\big|_{F_2}$.

After we have found w_0, v_0^1, and v_0^2, the following functions w_i, v_i^1, and v_i^2 are defined by induction. Suppose that for some i ($i \leq m$) all functions w_s, v_s^1, v_s^2 ($s < i$) have already been defined. Then, in the i-th equation (6) and in the conditions (11) and (16), the right-hand sides are already known. Under the assumptions made concerning the coefficients and the right-hand side of equation (1), the problems (6), (11), and (16) have a unique smooth solution.

Now define v_i^1. In equations (8) for v_i^1 and in the initial conditions (14), the right-hand sides are known functions. Obviously, the solutions of these problems, i.e. v_i^1, are functions of boundary-layer type near $x_0 = 0$.

Similarly, in equations (10) for v_i^2 and the boundary conditions (17), the right-hand sides are known functions. Then, solving these problems successively, we determine v_i^2 as a boundary-layer type function near the boundary F_2.

Multiply the functions v_i^1 and v_i^2 by the smoothing functions $\psi_1(x_0/\sigma)$ and $\psi_2(\rho/\sigma)$, respectively, and denote the new functions again by v_i^1 and v_i^2. Thus we have determined all the functions w_i, v_i^1, and v_i^2 entering the expansion (4).

It remains to estimate the remainder term Z_m. To this end, apply the operator $\mathcal{L}_\varepsilon$ to both parts of (4). We obtain

$$f = \mathcal{L}_\varepsilon u \equiv (\varepsilon \mathcal{L}_1 + \mathcal{L}_2)\left(w_0 + \sum_{i=1}^{m} \varepsilon^i w_i\right) + \left(M_0^1 + \sum_{i=1}^{m+1} \varepsilon^i M_i^1\right)\left(\sum_{i=0}^{m} \varepsilon^i v_i^1\right) +$$

$$+\left(M_0^2+\sum_{i=1}^{m+1}\varepsilon^i M_i^2\right)\left(\sum_{i=0}^{m}\varepsilon^i v_i^2\right)+\varepsilon^{m+1}\mathcal{L}_\varepsilon Z_m. \tag{19}$$

Taking into account the equations obtained by the iterative processes, from (19) we obtain

$$\mathcal{L}_\varepsilon Z_m = g(x_0, \mathbf{x}). \tag{20}$$

Thus, for the remainder term we obtain the following problem:

$$\mathcal{L}_\varepsilon Z_m = g(x_0, \mathbf{x}), \qquad Z_m\big|_{x_0=0} = \varphi_1, \qquad \frac{\partial Z_m}{\partial x_0}\bigg|_{x_0=0} = \varphi_2, \qquad Z_m\big|_F = \varphi_3.$$

Let us note that the functions φ_1, φ_2, and φ_3 satisfy the compatibility condition, i.e.

$$\varphi_1\big|_S = \varphi_3\big|_{x_0=0}, \qquad \varphi_2\big|_S = \varphi_3\big|_{x_0=0}.$$

We seek the function Z_m in the form

$$Z_m = Z_m^{(1)} + Z_m^{(2)},$$

where $Z_m^{(2)} = \varphi_1 + x_0\varphi_2 + \varphi_3 - (\varphi_1 + x_0\varphi_2)\big|_F$ (here it is assumed that the functions φ_i have already been multiplied by smoothing functions).

Obviously, $Z_m^{(1)}$ is the solution of the following problem:

$$\mathcal{L}_\varepsilon Z_m^{(1)} = g_1, \tag{21}$$

$$Z_m^{(1)}\big|_{x_0=0} = 0, \qquad \frac{\partial Z_m^{(1)}}{\partial x_0}\bigg|_{x_0=0}, \tag{22}$$

$$Z_m^{(1)}\big|_F = 0. \tag{23}$$

Lemma. *If the coefficients and the right-hand side of equation (21) are smooth functions and the operator $\mathcal{L}_2$ separates $\mathcal{L}_1$, then the estimate*

$$\|Z_m^{(1)}\| \leq M\|g_1\|,$$

holds, where M is a constant independent of ε, and the norm is understood in the sense of the metric of the space $\mathcal{L}_2(Q)$.

Let us note that a more general fact holds. If one considers the Cauchy problem for the hyperbolic equation $\mathcal{L}_\varepsilon u \equiv \varepsilon \mathcal{L}_1 u + \mathcal{L}_2 u = f$, where $\mathcal{L}_1$ is a hyperbolic operator of order $m+1$, $\mathcal{L}_2$ is a hyperbolic operator of order m, and the operator $\mathcal{L}_2$ separates $\mathcal{L}_1$, then the estimate

$$|D^{m-1}u, v| \leq M|f, v|,$$

holds, where v is the variable strip $0 \leq x_0 \leq t$;

$$|D^{m-1}u, v|^2 = \int_v \sum |D^\alpha u|^2 \, dv, \qquad \alpha \leq m+1.$$

Taking into account the lemma and the estimate

$$\|Z_m\| \leq \|Z_m^{(1)}\| + \|Z_m^{(2)}\|,$$

we assert that Z_m is bounded in the metric of $\mathcal{L}_2(Q)$.

Thus, the following has been proved.

Theorem. *If the coefficients and the right-hand side of equation (1) are smooth functions, $f(x_0, \mathbf{x})$ vanishes together with its derivatives up to the order needed by us along the characteristic $l(0,0)$, $a_0^0(\mathbf{x}) > 0$, and the operator $\mathcal{L}_2$ separates $\mathcal{L}_1$, then the solution of problem (1), (2), (3) is representable in the form (4), where the remainder term $\varepsilon^{m+1} Z_m$ tends to zero as $\varepsilon \to 0$ like ε^{m+1} in the sense of the metric of $\mathcal{L}_2(Q)$.*

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CITED LITERATURE

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