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Abstract

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ON THE APPLICATION OF DIFFERENCE SCHEMES WITH A SPLITTING OPERATOR TO HYPERBOLIC EQUATIONS WITH VARIABLE COEFFICIENTS

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The present work is devoted to the study of economical difference methods for equations of hyperbolic type with variable coefficients and is a direct continuation and strengthening of the results of works ([1–5]); the methodology used in it essentially coincides with the methodology of work ([4]). We note that works ([6,7]) are also devoted to a closely related circle of questions.

1. The initial problem is the following.

Problem 1. In the cylinder $Q_T = \overline{\Omega} \times [0 \leq x_0 \leq T]$, where $\overline{\Omega}$ is a closed domain in the space $x' = (x_1, x_2, \ldots, x_p)$, composed of a finite number of parallelepipeds with faces parallel to the coordinate planes, one seeks a solution of the equation

$$D_0^2 u = \sum_{k,s=1}^{p} D_k(a_{ks}(x) D_s u) + \sum_{s=1}^{p} (b_s(x) D_s u + c_s(x) u) + f(x), \qquad (1)$$

satisfying the initial and boundary conditions

$$u\,\Big|_{x_0=0} = \varphi(x'); \qquad D_0 u\,\Big|_{x_0=0} = \mu(x'); \qquad u\,\Big|_{S} = \psi(x), \quad x \in S. \qquad (2)$$

Here $D_s = \partial/\partial x_s$ $(s = 0, 1, \ldots, p)$, $x = (x_0, x')$, S is the lateral surface of Q_T,

$$a_{ks}(x) = a_{sk}(x), \qquad a_s = a_{ss}(x) \geq \gamma = \text{const} > 0$$

$$\left| \sum_{k \neq s} a_{ks} \xi_k \xi_s \right| \leq (1 - \sigma) \sum_{s=1}^{p} a_s \xi_s^2, \qquad (3)$$

$\sigma > 0$, $\xi = (\xi_1, \xi_2, \ldots, \xi_p)$ is an arbitrary real vector.

2. To construct a difference analogue of problem 1, introduce the notation: $i = (i_1, i_2, \dots, i_p)$ (i_s is an integer), $h > 0$ is the step of the spatial grid, $\{ih\}$ is the set of points of the spatial grid; accordingly the set of interior points Ω_h, the set of boundary points Γ_h, and $\overline{\Omega}_h = \Omega_h \cup \Gamma_h$ are defined; the value of the grid function v at the point $(n\tau, ih)$, where $\tau > 0$ is the step in x_0, is denoted by v_i^n,

$$v_i = v(ih), \qquad {}^{\pm s}i = (i_1, i_2, \dots, i_s \pm 1, \dots, i_p), \qquad {}^{\pm s}v_i = v({}^{\pm s}i\, h),$$

$$\Delta_s v_i = \frac{1}{h}({}^{+s}v_i - v_i), \qquad \overline{\Delta}_s v_i = \frac{1}{h}(v_i - {}^{-s}v_i), \qquad \widetilde{\Delta}_s v_i = \frac{1}{2}(\Delta_s + \overline{\Delta}_s) v_i,$$

$$(s = 1, 2, \dots, p),$$

$$\Delta_0 v_i^n = \frac{1}{\tau}(v_i^{n+1} - v_i^n), \qquad {}^{+\hat{s}}i = \left(i_1, \dots, i_s + \frac{1}{2}, \dots, i_p\right),$$

$${}^{+\hat{s}}a_i = a({}^{+\hat{s}}i\, h), \qquad t_n = n\tau.$$

$$L_s^n v_i = L_s v_i = \overline{\Delta}_s(a_s(t_n, ih)\, \Delta_s v_i) + b_s(t_n, ih)\, \widetilde{\Delta}_s v_i + c_s(t_n, ih)\, v_i, \qquad A^n v_i = A_s v_i = \left(E - \frac{\tau^2}{2} L_s\right) v_i,$$

E is the identity operator,

$$A v_i = \prod_{s=1}^{p} A_s v_i, \qquad B v_i^n = \left(2E + \tau^2 \sum_{k \neq s} \left(D_k a_{ks}^n \widetilde{\Delta}_s + a_{ks}^n \widetilde{\Delta}_k \widetilde{\Delta}_s\right)\right) v_i^n.$$

By the difference analogue of problem I, or **problem** I_h, we shall mean the problem of finding a grid function v satisfying the relation with the splitting operator A

$$\frac{1}{\tau^2} A\left(v_i^{n+1} + v_i^{n-1}\right) = \frac{1}{\tau^2} B v_i^n + f_i^n, \qquad ih \in \Omega_h \tag{4}$$

and the initial and boundary conditions

$$v_i^0 = \varphi_i, \qquad v_i^1 = g_i \qquad (ih \in \Omega_h),$$

$$v_i^k = \psi_i^k \qquad \left(ih \in \Gamma_h,\ k = 0, 1, \dots, \frac{T}{\tau}\right), \tag{5}$$

where the function g_i, expressed explicitly in terms of φ, μ, f, is such that $g_i = u_i^1 + O(\tau^q)$, where u is the exact solution of problem I, $q \geq 2$.

It can be verified that, for sufficiently smooth coefficients of equation (1), problem I_h, on the class of sufficiently smooth solutions of problem I, will approximate the latter in the metric of the space C with order of approximation $O(\tau^2 + h^2)$. Moreover, as follows from (5), the transition in problem I_h from one time layer to another requires $\sim 1/h^p$ arithmetic operations.

3. To formulate requirements on the coefficients of equation (1), let us introduce the following function spaces: H^0 is the space of functions bounded in Q_T:

$$\|u\|_{H^0} = \sup_{x \in Q_T} |u(x)|;$$

$H_l(m)$ $(l = 0, 1, \dots, p)$ is the space of functions having in Q_T bounded derivatives with respect to x_l of order not higher than m, satisfying the Lipschitz condition with respect to x_l; D_s^m $(s = 1, 2, \dots, p)$ is the space of functions having bounded derivatives in Q_T, containing no more than m differentiations with respect to each x_l $(l \neq 0,\ l \neq s)$. The norms in the enumerated spaces are introduced as sums of the H^0-norms of all derivatives entering into the definition of the corresponding space.

Theorem 1. *If conditions (3) are satisfied and*

$$a_s \in H_s(2), \qquad a_s \in D_s^1, \qquad b_s \in D_s^1, \qquad c_s \in D_s^1 \quad (s = 1, 2, \dots, p); \tag{6}$$

$$a_s \in H_0(0), \qquad D_k a_{ks} \in H^0, \qquad a_{ks} \in H_0(0) \quad (k \neq s), \qquad f \in H^0,$$

then there exist such $\tau_0 > 0$ and $h_0 > 0$ that, for all $\tau \leq \tau_0,\ h \leq h_0$, for the grid function y, which is a solution of problem I_h with zero boundary conditions $(\psi = 0)$, the difference a priori estimate

$$
\begin{aligned}
V(y^k, y^{k-1}) \equiv [(\Delta_0 y^{k-1})^2, 1] &+ \sum_{|\nabla|\le 1} [((\nabla y^k)^2 + (\nabla y^{k-1})^2), 1] \\
&+ \tau^2 \sum_{|\nabla|\le 2} [((\nabla y^k)^2 + (\nabla y^{k-1})^2), 1] \\
&+ \cdots + \tau^{2(p-1)} \sum_{|\nabla|\le p} [((\nabla y^k)^2 + (\nabla y^{k-1})^2), 1] \\
&\le M \left\{ \tau \sum_{n=1}^{k-1} [(f^n)^2, 1] + V(y^1, y^0) \right\},
\end{aligned} \tag{7}
$$

where $[z, w] = h^p \sum_{ih\in\bar{\omega}_h} z_i w_i$, $M > 0$ is a certain constant, and ∇y is a mixed difference with respect to the spatial variables of order $|\nabla|$, containing no more than one "differentiation" with respect to each spatial variable, $k \le T/\tau$.

The constants τ_0 and h_0 in Theorem 1 depend, first, on the dimensions of the domain Q_T and, second, on the norms of the coefficients in the corresponding spaces (conditions (6)). In a number of special cases the restrictions on the smallness of τ and h due to the second reason become superfluous. For example, if all $a_{ks} \equiv 0$ for $k \ne s$, or $a_s = \text{const}$, then the restriction on the smallness of h disappears; if the coefficient a_s does not depend on x_0, then the restriction on the smallness of τ disappears.

It follows obviously from Theorem 1 that, for $\tau \le \tau_0$, $h \le h_0$, problem I_h is well posed; here we note the fact that no restrictions on the smallness of τ/h were imposed.

Moreover, if the approximation conditions are satisfied, then for the difference of the solutions of problems I and I_h, $z = u - v$, using the a priori estimate (7), one can obtain

$$
V\left(z^k, z^{k-1}\right) = O\left(\tau^2 + h^2\right)^2 + \sum_{r=0}^{p-1} O\left(\frac{\tau^{q+r}}{h^{r+1}}\right)^2. \tag{8}
$$

In particular, if $q \ge 3$ and τ/h is bounded above, then

$$
V\left(z^k, z^{k-1}\right) = O\left(\tau^2 + h^2\right)^2. \tag{9}
$$

4. Convergence of the solutions of problem I_h to the solution of the problem with the rate of convergence (9) can be obtained under weaker conditions than the approximation conditions, if one refines estimate (7) for the case of a function f representable in "divergence" form. Namely, let

$$
f = F - \left(\frac{\tau^2}{2}\right)^3 \sum_{s_1,s_2,s_3,s_4} L_{s_1} L_{s_2} L_{s_3} L_{s_4} \Phi + \cdots + \left(-\frac{\tau^2}{2}\right)^{p-1} L_1 L_2 \cdots L_p \Phi, \tag{10}
$$

where $\sum_{s_1,s_2,s_3,s_4}$ denotes summation over all s_l $(l = 1,2,3,4)$, $s_1 < s_2 < s_3 < s_4$, $s_l = 1,2,\dots,p$, and the function Φ has in Q_T bounded derivatives with respect to all spatial variables, containing no more than one differentiation with respect to each x_s $(s = 1,2,\dots,p)$. Then the following theorem is valid:

Theorem 2. *Under the conditions of Theorem 1, for the mesh function y that is the solution of problem I_h with zero boundary conditions and right-hand side f representable in the form (10), there exist $\tau_0 > 0$, $h_0 > 0$ such that for all $\tau \le \tau_0$, $h \le h_0$ the a priori estimate holds*

$$V\left(y^k, y^{k-1}\right) \le M\left\{V\left(y^1, y^0\right) + \tau\sum_{n=1}^{k-1}\left\{\left[(F^n)^2, 1\right] + \tau^4\sum_{|\nabla|\le 4}\left[(\nabla\Phi^n)^2, 1\right] + \tau^6\sum_{|\nabla|\le 5}\left[(\nabla\Phi^n)^2, 1\right] + \dots + \tau^{2p-4}\sum_{|\nabla|\le p}\left[(\nabla\Phi^n)^2, 1\right]\right\}\right\}$$
$$\le M\left\{V\left(y^1, y^0\right) + \tau\sum_{n=1}^{k-1}\left[(F^n)^2, 1\right]\right\} + O\left(\tau^4\right). \tag{11}$$

If it is additionally assumed that $\frac{\tau^2}{h} \le l < \infty$ (l arbitrary), then the a priori estimate (11) can also be obtained in the case when

* Estimate (9) will also be valid for $q = 2$, if one assumes $D_s D_0^3 u \in H^0$ $(s = 1,2,\dots,p)$.

the function Φ from (10) has in Q_T bounded derivatives only of the form

$$D_{s_1}D_{s_2}D_{s_3}D_{s_4}\Phi(s_1 > s_2 > s_3 > s_4;\ s_k = 1,2,\dots,p;\ k = 1,2,3,4)^*.$$

5. A priori estimates of the type (7), (11) can be obtained for an entire class of difference schemes with a splitting operator, but their description is rather cumbersome. Therefore we indicate only a few cases.

For example, relation (4) may be replaced by any of the following:

$$\frac{1}{\tau^2}Av_i^{n+1} = \frac{1}{\tau^2}Bv_i^n - \frac{1}{\tau^2}\left(E - \frac{\tau^2}{2}\sum_{s=1}^{p}L_s\right)v_i^{n-1} + f_i^n, \tag{12}$$

$$\frac{1}{\tau^2}\left\{\bar{A}\left(v_i^{n+1} + v_i^{n-1}\right) - 2\bar{B}v_i^n\right\} = f_i^n; \tag{13}$$

where

$$\bar{A} = \prod_{s=1}^{p}\left(E - \frac{\tau^2}{2}L_s\right), \qquad \bar{B} = \bar{A} + \frac{\tau^2}{2}\sum_{s=1}^{p}L_s + \frac{1}{2}(B - 2E).$$

Moreover, in all the indicated schemes the operator L_s may be understood as the operator

$$L_s = \overline{\Delta}_s \left(\hat{a}_s \Delta_s\right) + b_s \widetilde{\Delta}_s + c_s E.$$

Then, naturally, in the operators $B, \bar{B}$ additional terms of the form

$$\tau^2 \sum_{s=1}^{p} \left\{ \overline{\Delta}_s \left((a_s - \hat{a}_s)\Delta_s\right) + (b_s - \tilde{b}_s)\widetilde{\Delta}_s + (c_s - \tilde{c}_s)E \right\}$$

must appear. In this case it is required that the coefficients a_s, b_s, c_s satisfy the same conditions (6) as the original coefficients, and that the coefficients $a_s - \hat{a}_s$, $b_s - \tilde{b}_s$, $c_s - \tilde{c}_s$ satisfy weaker conditions. It is also required that the coefficients $a_s - \hat{a}_s$ $(s = 1, 2, \dots, p)$ be sufficiently small.

Remark. Analogous difference schemes and a priori estimates have been obtained for a differential problem more general than Problem I. Namely, equation (1) may be replaced by

$$\rho(x) D_0^2 u = \sum_{k,s=1}^{p} D_k \left(a_{ks}(x) D_s u\right) + f(x, u, D_1 u, \dots, D_p u),$$

where $\rho(x) \geq \rho > 0$, $\rho(x) \in H_0(0)$, the coefficients a_{ks} satisfy conditions (3), and the function f has bounded partial derivatives with respect to $u, D_s u$ $(s = 1, 2, \dots, p)$.

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* In those cases when, for a difference problem written in the form $Lv = \mathcal{L}F$ (L and $\mathcal{L}$ are certain difference operators, F is a known function, v is the solution of the difference problem), it is possible to obtain an estimate analogous to (11) of the form $\|v\|_1 \leq M\|F\|_2$, it is apparently appropriate to say that the difference problem approximates the differential problem if $Lv - Lu = \mathcal{L}(f)$ (u is the solution of the differential problem) and $\|f\|_2 \to 0$ as the mesh size decreases.

Note: Figure translations are in progress. See original paper for figures.