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Abstract

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APPROXIMATION OF UNBOUNDED FUNCTIONS BY MODIFIED LANDAU AND BERNSTEIN POLYNOMIALS ON THE WHOLE PLANE

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1. In the present note asymptotic formulas are given which express the order of approximation by modified Landau and Bernstein polynomials of continuous unbounded functions defined in a multidimensional Euclidean space. For simplicity of exposition we shall consider only the case of two variables. It is not difficult to see that the results of this note can be extended to a larger number of variables.

2. We shall denote the whole plane by R_2, points of R_2 by $t \equiv (t_1, t_2)$, $x \equiv (x_1, x_2)$, and $\|x\| = \sqrt{x_1^2 + x_2^2}$. Let $T(\|x\|)$ be a monotonically increasing function: $T(\|x\|) \uparrow \infty$ $(\|x\| \to \infty)$.

We shall consider, on the whole plane, the modified Landau and Bernstein polynomials

$$L_n[f(\alpha_n t); \alpha_n^{-1} x] = \frac{n}{\pi} \int_E f(\alpha_n t) \prod_{i=1}^{2} \left[1 - (t_i - \alpha_n^{-1} x_i)^2\right]^n dt,$$

$$L_n^{(s)}[f(\alpha_n t); \alpha_n^{-1} x] = \frac{1}{n^{2s-1}\pi} \sum_{|\nu_i| \le n^s} f\left(\frac{\nu_1 \alpha_n}{n^s}, \frac{\nu_2 \alpha_n}{n^s}\right) \prod_{i=1}^{2} \left[1 - \left(\frac{\nu_i}{n^s} - \frac{x_i}{\alpha_n}\right)^2\right]^n,$$

$$B_n^{(1)}[f(\alpha_n t); \alpha_n^{-1} x] =$$

$$= \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} f\left(\frac{2\nu_1 - n_1}{n_1} \alpha_n, \frac{2\nu_2 - n_2}{n_2} \alpha_n\right) \prod_{i=1}^{2} \binom{n_i}{\nu_i} \frac{(1 + \alpha_n^{-1} x_i)^{\nu_i} (1 - \alpha_n^{-1} x_i)^{n_i - \nu_i}}{2^{n_i}},$$

$$B_n^{(2)}[f(\alpha_n t); \alpha_n^{-1}x] = \sum_{0 \le \nu_1+\nu_2 \le n} f\left(\left(\frac{2\nu_1}{n} - \frac{1}{2}\right)\alpha_n, \left(\frac{2\nu_2}{n} - \frac{1}{2}\right)\alpha_n\right) p_{\nu_1,\nu_2;n}(x),$$

where $\alpha_n \uparrow \infty$, $s \ge 1$, $n_1 + n_2 = n$, E is the square $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$, and

$$p_{\nu_1,\nu_2;n}(x) = \binom{n}{\nu_1, \nu_2} \frac{(1/2 + \alpha_n^{-1}x_1)^{\nu_1}(1/2 + \alpha_n^{-1}x_2)^{\nu_2}(1 - \alpha_n^{-1}x_1 - \alpha_n^{-1}x_2)^{n-\nu_1-\nu_2}}{2^n},$$

$$\binom{n}{\nu_1, \nu_2} = \frac{n!}{\nu_1!\nu_2!(n-\nu_1-\nu_2)!}.$$

Introduce the notation: $\Delta_n(\alpha_n) = L_n - f$, $\Delta_n^{(s)}(\alpha_n) = L_n^{(s)} - f$, $\delta_n^{(1)}(\alpha_n) = B_n^{(1)} - f$, and $\delta_n^{(2)}(\alpha_n) = B_n^{(2)} - f$.

Theorem 1. Let $\alpha_n \uparrow \infty$, $\beta_n \downarrow 0$ $(n \to \infty)$, and

$$1^\circ. \quad \alpha_n\beta_n \to 0, \qquad n\beta_n^4 \to 0 \quad (n \to \infty).$$

$$2^\circ. \quad \left(\frac{n}{\alpha_n}\right)^2 \exp\{T(\sqrt{2}\alpha_n) - 2n\beta_n^2\} \to 0 \quad (n \to \infty).$$

Then for every function $f(x) \in C^2(R_2)$, $f(x) = O(e^{T(\|x\|)})$ $(\|x\| \to \infty)$, the asymptotic equality

$$\Delta_n(\alpha_n) \sim \frac{\alpha_n^2}{4n}\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right) \qquad (n \to \infty) \tag{1}$$

holds.

Theorem 2. Let $s \ge 3/2$. Under the same assumptions as in Theorem 1, we have

$$\Delta_n^{(s)}(\alpha_n) \sim \frac{\alpha_n^2}{4n}\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right) \qquad (n \to \infty). \tag{2}$$

In particular, if we take $\alpha_n = n^\theta \left(0 < \theta < \frac{1}{m+2}\right)$, $T(\|x\|) = \|x\|^m$, and choose $\beta_n = n^{-\theta-\delta}$, where δ satisfies the conditions

$$\max(0,\, 1/4 - \theta) < \delta < 1/2[1 - (m+2)\theta],$$

then from Theorem 1 we obtain the following result.

Theorem 3. For every function $f(x) \in C^2(R_2)$, $f(x) = O(e^{\|x\|^m})$ $(\|x\| \to \infty)$, the asymptotic equality

$$\Delta_n(n^\theta) \sim \frac{1}{4}\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right)\left(\frac{1}{n}\right)^{1-2\theta} \qquad (n \to \infty). \tag{3}$$

If we take $\alpha_n = \log\log n$, then for a broader class of unbounded functions we obtain (choosing $\beta_n = \dfrac{1}{n^{1/4}\sqrt{\log n}}$) the following theorem.

Theorem 4. For every function $f \in C^2(R_2)$, $f(x) = O(e^{\|x\|})$ $(\|x\| \to \infty)$, we have

$$\Delta_n(\log\log n) \sim \frac{(\log\log n)^2}{4n}\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right) \qquad (n \to \infty). \tag{4}$$

If $s \geq 3/2$, then for the polynomials $L_n^{(s)}$ asymptotic formulas analogous to formulas (3) and (4) are valid. We shall not give them here.

Theorem 5. Let $\alpha_n \uparrow \infty$, $\beta_n \downarrow 0$ $(n \to \infty)$, and as $n \to \infty$:

1°. $\alpha_n\beta_n \to 0$.

2°. $n\exp\{T(\sqrt{2}\alpha_n) - 1/3\, n\beta_n^2\} \to 0$.

Then for every $f(x) \in C^2(R_2)$, $f(x) = O(e^{T(\|x\|)})$ $(\|x\| \to \infty)$, as $n \to \infty$ the asymptotic equality

$$\delta_n^{(2)}(\alpha_n) \sim \frac{\alpha_n^2}{2n}\left\{\left[\frac{1}{4} - \left(\frac{x_1}{\alpha_n}\right)^2\right]\frac{\partial^2 f}{\partial x_1^2} - 2\left(\frac{1}{2} + \frac{x_1}{\alpha_n}\right)\left(\frac{1}{2} + \frac{x_2}{\alpha_n}\right)\frac{\partial^2 f}{\partial x_1 \partial x_2} + \left[\frac{1}{4} - \left(\frac{x_2}{\alpha_n}\right)^2\right]\frac{\partial^2 f}{\partial x_2^2}\right\}. \tag{5}$$

Theorem 6. Let $\alpha_n \uparrow \infty$, $\beta_n \downarrow 0$ $(n \to \infty)$, and as $n_i \to \infty$ $(i = 1, 2)$:

1°. $\alpha_n\beta_n \to 0$.

2°. $n\exp\{T(\sqrt{2}\alpha_n) - 1/n_i\beta_n^2\} \to 0$.

Then for every function $f(x) \in C^2(R_2)$, $f(x) = O(e^{T(\|x\|)})$ $(\|x\| \to \infty)$, as $n \to \infty$ the asymptotic formula

$$\delta_n^{(1)}(\alpha_n) \sim \frac{\alpha_n^2}{2n_1}(1 - \alpha_n^{-2}x_1^2)\frac{\partial^2 f}{\partial x_1^2} + \frac{\alpha_n^2}{2n_2}(1 - \alpha_n^{-2}x_2^2)\frac{\partial^2 f}{\partial x_2^2} + \frac{\varepsilon_1\alpha_n^2}{n_1} + \frac{\varepsilon_2\alpha_n^2}{n_2}, \tag{6}$$

where $\varepsilon_1 \to 0$ as $n_1 \to \infty$ and $\varepsilon_2 \to 0$ as $n_2 \to \infty$.

From Theorem 5 there follow, in particular, the following results.

Theorem 7. For any function $f(x) \in C^2(R_2)$, $f(x) = O\big(e^{\|x\|^m}\big)$ as $\|x\| \to \infty$, as $n \to \infty$ we have

$$\delta_n^{(2)}(n^\theta) \sim \frac{1}{2}\left\{\left[\frac{1}{4} - \left(\frac{x_1}{n^\theta}\right)^2\right]\frac{\partial^2 f}{\partial x_1^2} - 2\left(\frac{1}{2} + \frac{x_1}{n^\theta}\right)\left(\frac{1}{2} + \frac{x_2}{n^\theta}\right)\frac{\partial^2 f}{\partial x_1 \partial x_2} + \left[\frac{1}{4} - \left(\frac{x_2}{n^\theta}\right)^2\right]\frac{\partial^2 f}{\partial x_2^2}\right\}\left(\frac{1}{n}\right)^{1-2\theta}. \quad (7)$$

Theorem 8. For any function $f(x) \in C^2(R_2)$, $f(x) = O\Big(e^{\|x\|^{\|x\|}}\Big)$ as $\|x\| \to \infty$, as $n \to \infty$ we have

$$\delta_n^{(2)}(\log\log n) \sim \frac{(\log\log n)^2}{2n}\left\{\left[\frac{1}{4} - \left(\frac{x_1}{\log\log n}\right)^2\right]\frac{\partial^2 f}{\partial x_1^2} - 2\left(\frac{1}{2} + \frac{x_1}{\log\log n}\right)\left(\frac{1}{2} + \frac{x_2}{\log\log n}\right)\frac{\partial^2 f}{\partial x_1 \partial x_2} + \left[\frac{1}{4} - \left(\right.\right.\right. \quad (8)$$

For the proof of Theorems 7 and 8, in Theorem 5 we put, respectively,

$$\beta_n = n^{-\theta-\delta}, \qquad \max(0,\ {}^1/_4 - \theta) < \delta < {}^1/_2\,[1-(m+2)\theta] \quad \text{and} \quad R = \frac{1}{n^{1/4}\sqrt{\log n}}.$$

From Theorem 6 one can obtain analogous results for Bernstein polynomials of the first type.

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Note: Figure translations are in progress. See original paper for figures.