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Abstract

Full Text

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On the Hydromagnetic Stability of a Rotating Plasma

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In the present work an equation is obtained that describes the propagation of linear waves in an ideally conducting compressible plasma cylinder with an arbitrary distribution of the internal magnetic field, velocity, density, and temperature, and some questions of stability with respect to perturbations developing along magnetic lines of force are considered.

Since linear waves of general form $\sim \exp i(kz - m\varphi - \omega t)$ in a coordinate system moving with the phase velocity of the wave $v_\phi = \omega/k$ are represented by stationary helical flows, the equation of small oscillations can be obtained starting from the equations of magnetohydrodynamics ([1]) for stationary flows

$$\rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p - \frac{1}{4\pi}[\mathbf{B}\ \mathrm{rot}\,\mathbf{B}] - \rho\nabla\Phi; \tag{1}$$

$$\mathrm{div}\,\rho\mathbf{v} = 0, \qquad \mathrm{div}\,\mathbf{B} = 0; \tag{2}$$

$$(\mathbf{v}\nabla)S = 0, \qquad \mathrm{rot}[\mathbf{v}\mathbf{B}] = 0, \tag{3}$$

where ρ is the density, p the pressure, S the entropy, $\mathbf{v}$ the velocity, $\mathbf{B}$ the magnetic field, and Φ the potential of the non-electromagnetic forces.

Under the assumption of helical symmetry of the flow, when in the cylindrical coordinate system r, φ, z all quantities depend only on r and $\theta = \varphi - \alpha z$, where $L = 2\pi/\alpha$ is the pitch of the helix, equations (1)–(3) can be reduced to a system of two equations for the functions ρ and ξ ([2]):

$$\Delta^*\xi + \frac{1}{2\beta\rho}\frac{\partial s}{\partial\xi}(\nabla\xi)^2 - \frac{\psi_0'^2}{\beta\rho^3}(\nabla\rho\,\nabla\xi) + \frac{1}{2\beta\rho}\frac{\partial}{\partial\xi}\frac{A^2}{s} +$$
$$+\frac{\beta}{2}\frac{\partial}{\partial\xi}\frac{B^2}{s} + \frac{\partial}{\partial\xi}\frac{AB\psi_0'}{\rho s\psi'} - \frac{2\alpha A}{\beta^2\rho} + TS' - U' = 0; \tag{4}$$

$$W(\rho, S) + \frac{v^2}{2} + \Phi + \frac{\beta B^2}{s} + \frac{AB\psi_0'}{\rho s \psi'} = U. \tag{5}$$

The streamlines of the fluid and the magnetic lines of force lie on the surfaces $\xi(r,\theta) = \text{const}$. The components of the velocity and magnetic field are expressed in terms of ξ by the following relations, which it is convenient to write in matrix form:

$$r\begin{pmatrix} v_r \\ H_r \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho}\psi_0' \\ \psi' \end{pmatrix} \frac{\partial \xi}{\partial \theta}, \qquad \alpha r \begin{pmatrix} v_z \\ H_z \end{pmatrix} - \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho}\psi_0 \\ \psi' \end{pmatrix} \frac{\partial \xi}{\partial r}, \tag{6}$$

$$\begin{pmatrix} v_z \\ H_z \end{pmatrix} + \alpha r \begin{pmatrix} v_\varphi \\ H_\varphi \end{pmatrix} = \frac{A}{s}\begin{pmatrix} \frac{1}{\rho}\psi_0' \\ \psi' \end{pmatrix} + \frac{\beta B}{s}\begin{pmatrix} \psi' \\ \psi_0' \end{pmatrix}.$$

The quantities $\psi_0, \psi, A, B, U,$ and S depend only on ξ, i.e., are constant on magnetic surfaces. A prime denotes differentiation with respect to ξ; the partial derivative with respect to ξ is taken at fixed ρ. By W and T are denoted the enthalpy and the temperature,

$$s = \frac{1}{\rho}\psi_0'^2 - \psi'^2, \qquad \beta = 1 + \alpha^2 r^2, \qquad \Delta^* = \frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\beta}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

$$\mathbf{B} = \sqrt{4\pi}\mathbf{H}.$$

In deriving (4)–(5), the thermodynamic equality

$$dW = \frac{1}{\rho}\, dp + T\, dS$$

was used.

Assuming that in the equilibrium state all quantities depend only on r, it is easy to obtain the relation between the equilibrium distributions of density, pressure, velocity, and magnetic field. Further putting $\Phi = 0$, we have

$$\frac{d}{dr}\left(p + \frac{H^2}{2}\right) + h_\varphi^2 - \rho v_\varphi^2 = 0, \tag{7}$$

where $rh_\varphi = H_\varphi, \quad rv_\varphi = v_\varphi$.

Choose the function ξ in the equilibrium state to be $\xi = r^2/2$; then, in the presence of a wave, $\xi = r^2/2 + \tilde{\xi}(r,\theta)$. The equilibrium density $\tilde{\rho}(r)$ also receives a certain perturbation $\tilde{\rho}(r,\theta)$, so that $\rho = \tilde{\rho}(r) + \tilde{\rho}(r,\theta)$. The relation between $\tilde{\rho}$ and $\tilde{\xi}$ is obtained from equation (5), if one uses the fact that $U = U(\xi)$. In the approximation linear in $\tilde{\rho}$ and $\tilde{\xi}$, putting $\tilde{\xi} = f(r)e^{im\theta}$, $\tilde{\rho} - (\rho'/r)\tilde{\xi} = g(r)e^{im\theta}$, and taking into account that $S = S(\xi)$, we find

$$g = -\frac{rs}{\beta G} f' - \frac{Q - b^2}{G} f, \tag{8}$$

where

$$G = c_T^2 s/\rho J_0^2 + H^2/\rho - r^2 J_0^2/\beta, \qquad Q = 2a/\beta + \rho v_\varphi^2 - h_\varphi^2,$$

$$s = \rho J_0^2 - J^2, \qquad a = \rho J_0 v_\varphi - J h_\varphi, \qquad J_0 b = J_0 h_\varphi - J v_\varphi,$$

$$J = aH_z - h_\varphi, \qquad J_0 = a(v_z - v_\Phi) - v_\varphi, \qquad \alpha = k/m, \qquad v_\Phi = \omega/k$$

is the phase velocity of the wave, and $c_T^2 = (\partial p/\partial \rho)_S = \gamma p/\rho$ is the square of the sound speed.

The equation for $\tilde{\xi}$ is obtained from (4) by linearization in $\tilde{\xi}$ and $\tilde{\rho}$, if one uses the dependence $U' = U'(\xi)$ and takes into account the equation of state and the adiabatic equation $p = \rho k T$, $p\rho^{-\gamma} = \exp \frac{\gamma - 1}{k} S$. Eliminating g with the aid of relation (8), we obtain as a result

$$\left(\frac{rs}{1-\chi}\frac{f'}{\beta}\right)' + \left\{-\frac{m^2 s}{r} + \frac{4\alpha^2 r a^2}{\beta s} - \left(\frac{Q - \chi b^2}{1-\chi}\right)' - \frac{\chi}{1-\chi}\frac{\beta}{rs}(Q - b^2)^2\right\} f = 0, \tag{9}$$

where

$$\chi = \frac{r^2 J_0^2}{\beta}\left(\frac{c_T^2 s}{\rho J_0^2} + \frac{H^2}{\rho}\right)^{-1}.$$

If the plasma is bounded by an ideally conducting shell located at $r = R$, then the boundary condition will be $f(R) = 0$. If, however, the plasma cylinder is held by an external magnetic field $\mathbf{B}_e = \sqrt{4\pi}\mathbf{H}_e$, then the boundary condition is the condition of pressure balance $p + H^2/2 = H_e^2/2$ on the perturbed surface of the cylinder $r = R + \delta R(\theta)$.

We shall describe the perturbed external field by the current function

$$\psi_e = \overline{\psi}_e + \widetilde{\psi}_e = \frac{\alpha r^2}{2} H_z^e - H_\varphi^e \ln r + f_e(r) e^{im\theta}.$$

The field components are determined through $\psi_e(r,\theta)$ by the relations

$$rH_r^e = \partial\psi_e/\partial\theta, \qquad \alpha r H_z^e - H_\varphi^e = \partial\psi_e/\partial r, \qquad H_z^e + \alpha r H_\varphi^e = \text{const}.$$

The radial part of the perturbation $f_e(r)$ satisfies the linear equation

$$\left(\frac{r}{\beta} f_e'\right)' - \frac{m^2}{r} f_e = 0, \tag{10}$$

whose solutions are the functions $rI_m'(amr)$ and $rK_m'(amr)$, where I_m and K_m are modified Bessel functions.

On the unperturbed surface of a plasma cylinder bordering on an external magnetic field, the boundary condition

$$\left\{\frac{rs}{1-\chi}\frac{f'}{f} + \beta\frac{Q-\chi b^2}{1-\chi} + rJ_e^2\frac{f_e'}{f_e} + \beta h_{\varphi e}^2\right\}_{r=R} = 0, \tag{11}$$

must be satisfied; it follows from pressure balance and the requirements $\xi|_{R+\delta R} = \text{const}$ and $\psi_e|_{R+\delta R} = \text{const}$. The quantities $J_e = \alpha H_z^e - h_{\varphi e}$, $h_{\varphi e} = H_\varphi^e/r$ are analogous.

introduced above for the internal field. The function $f_e(r)$ is chosen either from the requirement of decay as $r \to \infty$, if a plasma cylinder in free space is considered, or from the condition $f(R_e) = 0$, if at $r = R_e$ there is a perfectly conducting screen, etc.

Thus the ratio f_e'/f_e is a uniquely determined known function, and for f we have equation (9) with a boundary condition of the Sturm–Liouville type (11). According to relations (6), the radial part $f(r)$ of the perturbation $\xi(r,\theta)$ is proportional to the radial parts v_r and H_r, i.e., $f \sim rv_r/J_0 \sim rH_r/J$, and correspondingly $f_e \sim rH_r^e$.

Near the boundary $v_\Phi = 0$ of the stability region of a stationary plasma cylinder ($v_z = v_\varphi = 0$), the quantity $\chi \to 0$, and, except for the case $J = 0$, in equation (9) all terms $\sim \chi$ may be neglected. Further, $\chi \to 0$ as $c_T^2 \to \infty$ and as $H^2 \to \infty$. If in (9) and (11) terms of order χ may be neglected, we arrive at the equations

$$\left(\frac{rs}{\beta} f'\right)' + \left(-\frac{m^2 s}{r} + \frac{4\alpha^2 r a^2}{\beta s} + Q'\right) f = 0, \tag{12}$$

$$\left\{\frac{rs}{\beta}\frac{f'}{f} + Q + \frac{rJ_e^2}{\beta}\frac{f_e'}{f_e} + h_\varphi^2\right\}_{r=R} = 0, \tag{13}$$

which coincide with those obtained in ([3]) under the assumption of incompressibility of the fluid.

Since equation (9) describes waves of general helical type $\sim \exp i(kz - m\varphi - \omega t)$, then for $k \to 0$ and $k \to \infty$ it is easy to obtain from it the equations for azimuthal and axially symmetric waves, respectively.

We shall further restrict ourselves to the case $J = \alpha H_z - h_\varphi = 0$, or $\mathbf{kH} = 0$, when the perturbations are directed along the lines of force of the magnetic field. If the perturbations are helical, then, since $\alpha = \text{const}$, this case is realized for $\mu = h_\varphi / H_z = \text{const}$; for axially symmetric waves one must have $H = H_\varphi$, and for azimuthal waves $H = H_z$.

For $J = 0$, equation (9) is written in the form

$$\left(\frac{\rho r J_0^2 c_s^2 f'}{\beta c_s^2 - r^2 J_0^2}\right)' + \left\{-\frac{m^2 \rho J_0^2}{r} + \frac{4\alpha^2 r \rho v_\varphi^2}{\beta} + \left(\frac{\beta c_s^2 Q - r^2 J_0^2 h_\varphi^2}{\beta c_s^2 - r^2 J_0^2}\right)' - \frac{r\beta \left(Q - h_\varphi^2\right)^2}{\rho \left(\beta c_s^2 - r^2 J_0^2\right)}\right\} f = 0, \tag{14}$$

where $c_s^2 = c_T^2 + H^2/\rho$. Let us consider several cases in which from equation (14) one can simply obtain the conditions for the onset of instability, i.e., the conditions for the existence of solutions with frequency ω containing an imaginary part.

1. A. Let the plasma rotate as a whole* ($v_\varphi = \text{const}$), so that $J_0 = -\alpha v_\Phi - v_\varphi = \text{const}$; then, as $J_0 \to 0$, according to (14), we have

$$\left(\frac{\rho r}{\beta} f'\right)' + \left\{-\frac{m^2 \rho}{r} + \frac{1}{J_0^2}\left[\frac{4\alpha^2 r \rho v_\varphi^2}{\beta} + \left(\rho v_\varphi^2 - h_\varphi^2\right)' - \frac{r\left(\rho v_\varphi^2 - 2h_\varphi^2\right)^2}{\rho c_s^2}\right]\right\} f = 0. \tag{15}$$

It follows from this ([4]) that a necessary condition for stability ($J_0^2 > 0$) of a plasma cylinder bounded by a perfectly conducting shell, with respect to helical waves, is the positivity of the expression in square brackets:

$$\frac{4\alpha^2 r \rho v_\varphi^2}{\beta} + \left(\rho v_\varphi^2 - h_\varphi^2\right)' - \frac{r\left(\rho v_\varphi^2 - 2h_\varphi^2\right)^2}{\rho c_s^2} > 0. \tag{16}$$

B. If a plasma cylinder with a homogeneous longitudinal current ($h_\varphi = \text{const}$) rotates as a whole ($v_\varphi = \text{const}$), then for long-wavelength oscillations $\alpha^2 r^2 \ll 1$, in the case $v_\varphi^2 \ll c_s^2$, equation (14) takes the form

$$(\rho r f')' + \left\{-\frac{m^2 \rho}{r} - \frac{\rho'}{J_0^2}\left(2J_0 v_\varphi + v_\varphi^2\right)\right\} f = 0. \tag{17}$$

* Here this restriction is not essential.

Let us consider a cylinder with a free boundary and without surface currents; then $J_e(R) = 0$, and the boundary condition (11) is written in the form

$$\left\{ J_0^2 \frac{rf'}{f} + 2J_0 v_\varphi + v_\varphi^2 \right\}_{r=R} = 0. \tag{18}$$

Equation (17) and the boundary condition (18) are satisfied by the solution $f \sim r^m$, whence for the frequency ω we obtain the expression $\omega = (1-m \pm \sqrt{1-m})v_\varphi$. If there is no longitudinal current, then the case $J = 0$ corresponds to azimuthal waves. This case was considered in Ref. ([5]), and for a particular density distribution $\rho \sim \exp(-qr^2)$, in Ref. ([6]). For $\rho = \text{const}$, equation (14) is solved exactly ([3]), and for long-wavelength oscillations we obtain, naturally, the same formula for ω, since it does not contain ρ. Thus, for an arbitrary dependence $\rho(r)$, the rotating plasma column as a whole with a free boundary is hydrodynamically unstable with respect to perturbations parallel to **H**. The increment of the instability development $\sqrt{m-1}\, v_\varphi$ grows with m, which indicates a tendency toward unstranding of the rotating plasma.

2. For axially symmetric waves in a rotating plasma, when $H = H_\varphi$, equation (14) takes the form

$$\left(\frac{\rho c_s^2}{c_s^2 - v_\Phi^2} \frac{f'}{r} \right)' + \left\{ -\frac{k^2 \rho}{r} + \frac{4\rho v_\varphi^2}{r v_\Phi^2} + \left[\frac{c_s^2(\rho v_\varphi^2 - h_\varphi^2) - v_\Phi^2 h_\varphi^2}{v_\Phi^2 (c_s^2 - v_\Phi^2)} \right]' - \frac{r(\rho v_\varphi^2 - 2h_\varphi^2)^2}{\rho v_\Phi^2 (c_s^2 - v_\Phi^2)} \right\} f = 0. \tag{19}$$

As $v_\Phi^2 \to 0$, from it one obtains an equation of type (15), whence follows the necessary condition for stability

$$\frac{4\rho v_\varphi^2}{r} + (\rho v_\varphi^2 - h_\varphi^2)' - \frac{r(\rho v_\varphi^2 - 2h_\varphi^2)^2}{\rho c_s^2} > 0. \tag{20}$$

The sufficiency of this condition for stability with respect to axially symmetric perturbations was established in Ref. ([7]). For a nonrotating plasma ($v_\varphi = 0$), condition (20) coincides with the necessary condition for stability of a plasma with closed field lines, which follows from Kadomtsev' s criterion ([8]):

$$-w \left(\frac{\nabla u}{u} \right)^2 < \nabla u \nabla p < \gamma p \frac{(\nabla u)^2}{|u|},$$

where

$$u = -\oint \frac{dl}{H}, \qquad w = \oint H\, dl$$

for $H = H_\varphi$.

3. For azimuthal waves $\left(J_0 = -\frac{\omega}{m} - v_\varphi\right)$, when $H = H_z$, equation (14) gives

$$\left(\frac{\rho r J_0^2 c_s^2 f'}{c_s^2 - r^2 J_0^2}\right)' + \left\{-\frac{m^2 \rho J_0^2}{r} + \left[\frac{c_s^2(2\rho J_0 v_\varphi + \rho v_\varphi^2)}{c_s^2 - r^2 J_0^2}\right]' - \frac{r(2\rho J_0 v_\varphi + \rho v_\varphi^2)^2}{\rho(c_s^2 - r^2 J_0^2)}\right\} = 0. \tag{21}$$

Letting J_0 tend to zero, analogously to the preceding case, we obtain the stability criterion with respect to azimuthal (flute) perturbations in the form

$$(\rho v_\varphi^2)' - \frac{\rho r v_\varphi^4}{c_s^2} > 0. \tag{22}$$

It follows from this that, in the case of uniform rotation ($v_\varphi = \text{const}$), stability requires $\rho' > 0$.

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Note: Figure translations are in progress. See original paper for figures.