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Abstract

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ON THE POINCARÉ–ANDRONOV METHOD IN THE PROBLEM OF PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DELAY

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The paper considers a system of differential equations (in vector notation)

$$x'(t) = f(t, T_t x), \tag{1}$$

in which the derivative $x'(t)$ depends on the "prehistory" $x(\tau)$, $t - h \leq \tau \leq t$, and the right-hand side is ω-periodic.

The method of the Poincaré–Andronov point transformation is extended to equations (1) for arbitrary h and ω. A method is justified for continuing a periodic solution with respect to the delay parameter h from an ordinary system of differential equations.

1. Let us describe equation (1) in greater detail. Let $0 < h < \infty$, and let $M[-h, 0]$ be the set of piecewise continuous (i.e., on a finite interval having no more than a finite number of points of discontinuity, at which finite left and right limits exist) vector functions $x(s) = (x_1(s), \dots, x_n(s))$, $-h \leq s \leq 0$; the set $M[-h, \omega]$ is defined similarly. Define the operator $T_t x = x(t+s)$, $-h \leq s \leq 0$, acting from any $M[-h, t]$, $t \geq 0$, into $M[-h, 0]$. Let

$$f(t, x) = (f_1(t, x), \dots, f_n(t, x)),$$

where $f_i(t, x)$, for almost every t, $-\infty < t < \infty$, is a functional on $M[-h, 0]$, and for every function $x \in M[-h, \omega]$ the function $f(t) = f(t, T_t x)$ is absolutely summable on $[0, \omega]$. Assume also that $f(t, x)$ is ω-periodic in t, i.e., for almost all t and all x, $f(t + \omega, x) \equiv f(t, x)$.

We pose the Cauchy problem for equation (1):

$$T_t x\big|_{t=0} = x^0, \quad \text{where } x^0 \in M[-h, 0]. \tag{2}$$

By a solution of problem (1)–(2) we shall mean a function $x \in M[-h, l]$, $l > 0$, absolutely continuous and satisfying equation (1) for $t \geq 0$ and condition (2).

Equations of the form (1) were considered by N. N. Krasovskii ([1]). As an example we give the equation

$$x'(t) = Ax(t) + Bx(t - h(t)) + \int_{t-h(t)}^{t} \Phi(t, s; x(t), x(s))\, ds, \tag{3}$$

where $h(t)$ is an ω-periodic continuous function.

2. **The point-transformation operator.** Equation (1) with delay parameter h may be regarded as an equation with parameter $\bar{h} \geq h$. In general, we shall call two equations of type (1) **equivalent** if their solutions for $t \geq 0$ coincide when the initial conditions coincide on the common part of their domains of definition. Therefore

one may assume that $h = k\omega$ (k—an integer or ∞); we shall call such a parameter **normalized**.

Denote by $C(-\omega, 0]$ the space of uniformly continuous functions $x^0(s)$ on the interval $-\omega < s \leq 0$, with the usual norm $\|x^0\| = \sup |x^0(s)|$. By $C^k(-\omega, 0]$ we denote the set of initial functions for problem (1)–(2) of the form

$$x(\tau) = x^0(\tau + i\omega) \quad \text{for } -(i+1)\omega < \tau \leq -i\omega, \qquad (i = 0, 1, \dots, k-1).$$

Suppose that the solution $x(t, \tilde{x})$ with initial condition $\tilde{x}$ is unique and is defined up to $t = \omega$. Then in the space $C(-\omega, 0]$ the operator

$$\Pi_\omega x^0 = x(\omega + s, \tilde{x}), \qquad -\omega < s \leq 0, \tag{4}$$

is defined; its significance for periodic solutions is asserted by the following theorem:

Theorem 1. In order that the solution $x(t)$ with initial condition $\tilde{x}$ be ω-periodic, it is necessary and sufficient that the function $x^0(s)$ be a fixed point of the operator Π_ω: $\Pi_\omega x^0 = x^0$.

Let us show that the operator (4) is completely continuous. Indeed, let the conditions

$$\left| \int_0^t f(\tau, T_\tau x)\, d\tau \right| \leq M(r), \qquad 0 \leq t \leq \omega; \tag{5}$$

$$\int_0^t f(\tau, T_\tau x)\, d\tau \quad \text{are equicontinuous}, \qquad 0 \leq t \leq \omega, \tag{6}$$

be fulfilled for any bounded set of functions $\|x\| \le r,\ x \in M[-k\omega, \omega]$,

$$\int_0^t f(\tau, T_\tau x^m)\, d\tau \to \int_0^t f(\tau, T_\tau x)\, d\tau, \qquad 0 \le t \le \omega, \tag{7}$$

if $\|x^m - x\| \to 0,\ x^m, x \in M[-k\omega, \omega]$.

Theorem 2. Suppose that problem (1)–(2) has a unique solution on the interval $[-k\omega, \omega]$ for any choice of the initial condition (2), and that conditions (5)–(7) are fulfilled; suppose that Π_ω is a bounded operator (i.e., transforms every bounded set into a bounded set).

Then the operator Π_ω acts completely continuously in the space $C(-\omega, 0]$.

Let us note that in the case $h \le \omega$ the complete continuity of the operator Π_ω was established by A. Halanay [2], who studied it in connection with periodic solutions. Below, for the study of fixed points of the operator Π_ω, we shall apply the rotation of the vector field [3].

3. Continuation of periodic solutions with respect to the lag parameter h**.** Suppose that $h = h(\lambda)$ depends on the numerical parameter λ, $h(1) = k\omega$, and for all λ, $0 \le \lambda \le 1$, $h(\lambda) \le k\omega$.

The functional $f(\lambda, t, x)$ in this case depends on λ; its lag parameter may not be normalized for $\lambda \ne 1$. Introduce another lag parameter $\bar{h} = k\omega$. If the functional $f(\lambda)$ is regarded as a functional $\bar{f}(\lambda)$ with lag parameter (already normalized for all λ) $\bar{h}$, then equations (1) and

$$x' = \bar{f}(\lambda, t, T_t x) \tag{1'}$$

are equivalent, and it suffices to study periodic solutions of equation $(1')$.

Thus, the problem of continuation of a periodic solution with respect to the parameter h is equivalently reduced to the study of the dependence of a periodic solution of equation $(1')$ on the parameter λ.

Under certain conditions ((5)—(7), etc.) with respect to equation $(1')$, the operator (4), depending on λ, $\Pi_\omega(\lambda)$, is quite continuous in the aggregate (λ, x^0). Therefore, for the study of the fixed points of $\Pi_\omega(\lambda)$ one may apply the following general scheme ([3],[4]): for all λ, $0 \leqslant \lambda \leqslant 1$, an a priori estimate is known for the fixed points of the operator $\Pi_\omega(\lambda)$; for $\lambda = 0$ it is possible to compute the rotation of the field $x^0 - \Pi_\omega(0)x^0$ on a sufficiently large sphere in $C(-\omega, 0]$; let it be different from zero; then for all $0 \leqslant \lambda \leqslant 1$ there exists a fixed point.

For $\lambda = 0$ the rotation of the field may turn out to be simply computable. For concrete equations this will be the case, for example, when for $\lambda = 0$ the function $h(t)$ is equal to zero or has the property $h(0) = 0$. Then the rotation of the field $x^0 - \Pi_\omega(0)x^0$ is computed through its finite-dimensional component. For example, replacing in (3) the function $h(t)$ by the function $\lambda h(t)$, $0 \leqslant \lambda \leqslant 1$, we obtain, for $\lambda = 0$, an equation equivalent to the ordinary equation $x' = (A+B)x$. We note that concrete equations with delay parameter $h(t)$, under the condition $t - h(t) \geqslant 0$, have been studied by a number of authors (see the survey ([5])).

4. We shall illustrate the method described above by one theorem on the existence of a second periodic solution. Consider equation (1), depending on the parameters μ, λ:

$$x' = f(\mu, \lambda, t, T_t x), \qquad -P \leqslant \mu, \lambda \leqslant P, \tag{8}$$

with delay parameter $h = k\omega$. Suppose that conditions (5), (6) are satisfied uniformly for all (μ, λ), and the condition

$$\int_0^t f(\mu^m, \lambda^m, \tau, T_\tau x^m)\, d\tau \to \int_0^t f(\mu, \lambda, \tau, T_\tau x)\, d\tau, \qquad 0 \leqslant t \leqslant \omega, \tag{7'}$$

when $\mu^m \to \mu,\ \lambda^m \to \lambda,\ \|x^m - x\| \to 0,\ x^m, x \in M[-k\omega, \omega]$. Let the inequality

$$|f(\mu, \lambda, t, x) - f(\mu, \lambda, t, y)| \leqslant M(t)\|x - y\|, \qquad 0 \leqslant t \leqslant \omega, \tag{9}$$

be satisfied, where $x, y \in M[-k\omega, 0]$ and the norm is taken in this space, $M(t) \geqslant 0$ is summable on $[0, \omega]$. Suppose that for $\lambda = 0$ the equalities

$$f(\mu; 0, t, x) = A_i(\mu, t, x) + \varphi_i(\mu, t, x) \qquad (i = 1, 2), \qquad -p \leqslant \mu \leqslant p, \tag{10}$$

hold, where $p \leqslant P$, and A_i are ω-periodic in t, linear in x functionals satisfying the inequalities

$$|A_i(\mu, t, x)| \leqslant M_i(t)\|x\| \qquad (i = 1, 2). \tag{11}$$

Here $M_i \geqslant 0$ and are summable on $[0, \omega]$; let A_1 and A_2 satisfy conditions (5), (6) uniformly in μ, and a condition of type $(7')$ with $\lambda = 0$. Let the equations

$$x' = A_i(\mu, t, T_t x) \qquad (i = 1, 2) \tag{12}$$

for all $\mu,\ -p \leqslant \mu \leqslant p$, have only the zero ω-periodic solution and, for $\mu = 0$, be equivalent to the ω-periodic linear ordinary differential equations

$$x' = A_i^*(t)x \qquad (i = 1, 2). \tag{13}$$

With respect to the latter, assume that they are reducible, by means of a real ω-periodic nonsingular transformation, to equations with constant matrices.

We impose conditions on the functionals φ_1 and φ_2. Let

$$|\varphi_i(t,x)| \le L(t)O_i(\|x\|), \qquad 0 \le t \le \omega \quad (i=1,2), \tag{14}$$

where $L(t) \ge 0$ is summable on $[0,\omega]$, $O_i(u)$ are monotonically increasing functions of u, $0 \le u < \infty$, and $\frac{1}{u}O_1(u) \to 0$ as $u \to 0$, $\frac{1}{u}O_2(u) \to 0$ as $u \to \infty$. We note that for φ_1 it is sufficient that (14) hold for small x.

Theorem 3. Suppose the preceding conditions are satisfied. Suppose all characteristic exponents β of equations (13) are nonzero, and in the case $\beta = \pm ir$ (purely imaginary) the condition $r < 2\pi\omega^{-1}$ is satisfied. Suppose the numbers of positive characteristic exponents (counting their multiplicities) for each of equations (13) have different parity.

Then there exist two numbers $\delta_1 > 0$, $\delta_2 > 0$ such that, for all μ, λ, $|\mu| \le p + \delta_1$, $|\lambda| \le \delta_2$, equation (8) has at least two ω-periodic solutions ($\delta_1 = 0$ in the case $p = P$).

As an example, consider the equation

$$x' = \varphi[t, x(t), x(t-h(t))] + \lambda \int_{t-h(t)}^{t} K(t, s-t)\psi[t, x(t), x(s)]\, ds, \tag{15}$$

where $\varphi(t,x,u)$, $\psi(t,x,u)$ are continuous n-dimensional vector functions, $-\infty < t < \infty$, x, u belong to n-dimensional space, ω-periodic in t and satisfying a Lipschitz condition in (x,u), with

$$|\varphi(t,x,u) - A_1 x - B_1 u| \le a(|x| + |u|)^{1-\varepsilon} + b, \qquad a, b > 0,$$

$$|\varphi(t,x,u) - A_2 x - B_2 u| \le c(|x| + |u|)^{1+\gamma}, \qquad |x| + |u| < \delta.$$

Here A_i, B_i are constant matrices, with $A_i + B_i$ nonsingular, and their numbers of positive eigenvalues β have different parity; in the purely imaginary case $\beta = \pm ir$ suppose $r < 2\pi\omega^{-1}$. The square matrix kernel $K(t,s)$ is continuous in (t,s) and ω-periodic in t; $h(t)$ is continuous and ω-periodic. If the equations

$$x' = A_i x(t) + B_i x[t - \mu h(t)] \qquad (i = 1, 2)$$

for all μ, $0 \le \mu \le 1$, have only the zero ω-periodic solution, then equation (15), for small λ, has at least two ω-periodic solutions.

In conclusion we mention a number of works [(6–8)] devoted to the method of the point transformation. The problem of the topological study of this transformation for arbitrary h, ω was posed at M. A. Krasnosel' skii' s seminar. He reported that results close to ours were obtained by the method of integral equations.

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Note: Figure translations are in progress. See original paper for figures.