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Abstract

Full Text

Mathematics

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On Congruent and Asymptotic Differentiability

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1. Let $f(x)$ be a real function defined on the interval (a, b), let E be some set of points x of the interval (a, b), and let Q be a set of points h having zero as its limit point.

We shall call the limit

$$f'_Q(x) = \lim_{h\to 0,\ h\in Q} \frac{f(x+h) - f(x)}{h}$$

the Q-derivative of the function $f(x)$ at the point $x \in E$.

In what follows we shall assume that the set Q is chosen the same for all values x under consideration ($x \in E$). In the case where the finite Q-derivative $f'_Q(x)$ exists for all $x \in E$, we shall say that $f(x)$ is Q-differentiable on E, or differentiable on E with respect to congruent sets. The upper and lower Q-derivative numbers of the function $f(x)$ are defined as follows:

$$\overline{f}'_Q(x) = \overline{\lim}_{h\to 0,\ h\in Q} \frac{f(x+h) - f(x)}{h}, \qquad \underline{f}'_Q(x) = \underline{\lim}_{h\to 0,\ h\in Q} \frac{f(x+h) - f(x)}{h}.$$

In our papers ([3,4]) we found a condition (necessary and sufficient) that must be imposed on the set Q in order that from the Q-differentiability of $f(x)$ (or from the finiteness of the Q-derivative numbers) on a set E of positive measure it should follow that ordinary differentiability holds almost everywhere on E. In the class of measurable functions $f(x)$ this condition is as follows:

$$\lim_{\delta\to 0+} \frac{\operatorname{mes} Q \cdot (-\delta, \delta)}{2\delta} > 0;$$

for continuous functions, however, it is substantially weaker:

$$\lim_{\delta\to 0+} \frac{\operatorname{mes} \overline{Q} \cdot (-\delta, \delta)}{2\delta} > 0$$

(here $\overline{Q}$ is the closure of Q). We now pose the following question.

What must the set Q be (a necessary and sufficient condition) in order that Q-differentiability of a function $f(x)$ (or even the fulfillment of one of the inequalities $\overline{f}'_Q(x) < +\infty$, $\underline{f}'_Q(x) > -\infty$) at every point of a set of positive measure imply asymptotic differentiability of $f(x)$ almost everywhere on this set.

Definition. We shall say that a set Q belongs to the class (D) if from it one can select a sequence of points $h_n \in Q$,

$$\lim_{n\to\infty} h_n = 0,$$

such that

$$\lim_{n\to\infty} \left|\frac{h_{n+1}}{h_n}\right| > 0$$

*.

* A set Q belonging to the class (D) may be such that neither its part lying to the right of zero nor its part lying to the left of zero, taken separately, belongs to the class (D).

A set Q not belonging to the class (D) may nevertheless have upper density at zero equal to 1:

$$\lim_{\delta\to 0+} \frac{\operatorname{mes} Q \cdot (-\delta, \delta)}{2\delta} = 1,$$

i.e. be sufficiently "dense."

Theorem 1. Let $f(x)$ be a measurable function on the interval (a, b), and let the set Q belong to class (D). Then the condition $\overline{f}'_Q(x) < +\infty$, or $\underline{f}'_Q(x) > -\infty$, at every point x of a set $E \subset (a, b)$ of positive measure entails the asymptotic differentiability of $f(x)$ almost everywhere on this set.

For every set Q not belonging to class (D), however, one can construct a function $f(x)$, continuous on the interval (a, b), which has a finite Q-derivative and has no asymptotic derivative at every point of some set $\mathcal{E} \subset (a, b)$ of positive measure (the measure of $\mathcal{E}$ can be made arbitrarily close to $b - a$).

As is seen from Theorem 1, the solution of the question posed turns out to be the same both for the class of measurable functions and for the class of continuous functions.

We note that the requirement of measurability of $f(x)$ in the first part of Theorem 1 cannot be removed for every set Q of class (D). There exist sets Q of class (D) for which this requirement is necessary. Such sets may even be countable sequences. On the other hand, one can define sets Q of class (D) for which the requirement of measurability of $f(x)$ is superfluous. (For example, Q is an interval containing 0, or an interval with endpoint at 0.)

It is interesting to note the circumstance that, for sequences $\{h_n\}$ tending rapidly to 0, the existence of a finite limit of the ratio

$$\frac{f(x+h_n)-f(x)}{h_n}$$

(as $h_n \to 0$; $\{h_n\}$ does not depend on x) on a set of positive measure does not entail even the asymptotic differentiability of $f(x)$ anywhere on this set; for example, in the case when $\{h_n\} = Q$ tends to 0 faster than a geometric progression, i.e.

$$|h_{n+1}| = o(|h_n|).$$

2. It is known that the ordinary derivative $f'(x)$ cannot be equal to infinity (of a definite sign) at every point of a set of positive measure. This was proved by N. N. Luzin ([1,2]). The ordinary derivative may be regarded as a Q-derivative $f'_Q(x)$ with respect to congruent sets, taking for Q any interval containing 0. It is natural to pose the question under what conditions (concerning the structure of the set Q) the equality $f'_Q(x) = +\infty$ is possible on a set E (of points x) of positive measure, and under what conditions this cannot occur.* A complete solution of this question is given by the following theorem.

Theorem 2. Let the function $f(x)$ be measurable on (a,b), and let the set Q belong to class (D). Then the set $E \subset (a,b)$ of all points x where the equality

$$f'_Q(x) = +\infty$$

holds has measure 0.

For every set Q not belonging to class (D), one can construct a function $f(x)$, continuous on the interval (a,b), for which

$$f'_Q(x) = +\infty$$

on some set $\mathcal{E}$ of positive measure (the measure of $\mathcal{E}$ can be made arbitrarily close to $b-a$).

The first part of this theorem generalizes the result of N. N. Luzin mentioned above.

In Theorem 2, just as in Theorem 1, we obtain the same solution of the question both in the class of measurable functions and in the class of continuous functions. The requirement of measurability of $f(x)$ cannot be removed for all sets Q of class (D).

Since, according to the first part of Theorem 2, for the lower Q-derivative number the equality $f'_Q(x) = +\infty$ is possible only on a set of measure zero (if Q

is of class (D)), the condition $\overline{f}'_Q(x) < +\infty$, or $\underline{f}'_Q(x) > -\infty$, in Theorem 1 is equivalent to the requirement that at least one of the Q-derivative numbers be finite.

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* The equality $f'_Q(x) = +\infty$ in this problem can, of course, be replaced by $f'_Q(x) = -\infty$ or by $|f'_Q(x)| = +\infty$.

According to the second part of Theorem 2, for every sequence $\{h_n\}$ that forms a set not belonging to the class (D), in particular one tending to zero faster than a geometric progression, there exists a continuous function $f(x)$ with difference quotient tending to $+\infty$ on a set of positive measure:

$$\lim_{h_n \to 0} \frac{f(x+h_n) - f(x)}{h_n} = +\infty$$

(we emphasize the independence of $\{h_n\}$ from x).

3. In conclusion we note that for Q-derivative numbers the theorem analogous to the theorem of N. N. Luzin–A. Denjoy on ordinary derivative numbers is false (at least for sets Q not belonging to the class (D)).

According to the theorem of N. N. Luzin–A. Denjoy, the upper and lower (for example, right-hand) derivative numbers of Dini cannot be finite and distinct at every point of a set of positive measure. However, for Q-derivative numbers such a phenomenon is possible.

Theorem 3. *If the set Q does not belong to the class (D), then one can construct a continuous function $f(x)$ on the interval (a, b) which at every point of some set $\mathcal{E} \subseteq (a, b)$ of positive measure will have unequal finite Q-derivative numbers*

$$-\infty < \underline{f}'_Q(x) < \overline{f}'_Q(x) < +\infty$$

(for example, $\overline{f}'_Q(x) = 1$, $\underline{f}'_Q(x) = -1$) and therefore will not have the Q-derivative $f'_Q(x)$. The measure of the set $\mathcal{E}$ may be made arbitrarily close to $b - a$.

Theorem 3 remains valid if in its formulation the condition

$$-\infty < \underline{f}'_Q(x) < \overline{f}'_Q(x) < +\infty$$

is replaced by any of the following conditions:

1)
$$-\infty = \underline{f}'_Q(x) < \overline{f}'_Q(x) = +\infty;$$

2)

$$-\infty < \underline{f}'_Q(x) < \overline{f}'_Q(x) = +\infty;$$

3)

$$-\infty = \underline{f}'_Q(x) < \overline{f}'_Q(x) < +\infty.$$

*

With respect to sets Q from the class (D), we can say only the following:

If Q additionally satisfies the condition

$$\lim_{\delta\to 0+} \frac{\operatorname{mes} Q \cdot (-\delta, \delta)}{2\delta} > 0,$$

and $f(x)$ is a measurable function (or

$$\lim_{\delta\to 0+} \frac{\operatorname{mes} \overline{Q} \cdot (-\delta, \delta)}{2\delta} > 0$$

in the case of continuous $f(x)$), then the condition

$$-\infty < \underline{f}'_Q(x) < \overline{f}'_Q(x) < +\infty$$

on a set of positive measure entails Q-differentiability almost everywhere on this set (and even $f'_Q(x) = f'(x)$).

This follows from Theorems 2 and 4 of paper ([3]).

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CITED LITERATURE

1. N. N. Luzin, *Matem. sborn.*, **28**, no. 2, 266 (1912).

2. N. N. Luzin, *Integral and Trigonometric Series*, 1951, p. 278.

3. G. Kh. Sindalovskii, *DAN*, **134**, no. 6, 1305 (1960).

4. G. Kh. Sindalovskii, *Izv. AN SSSR, Ser. matem.*, **26**, 125 (1962).

* As follows from Theorem 2, for sets Q not belonging to the class D, the equalities $f'_Q(x) = +\infty$ or $f'_Q(x) = -\infty$ are also possible on a set of positive measure.

Note: Figure translations are in progress. See original paper for figures.