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1. Consider the implicit difference scheme:

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Abstract

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MATHEMATICS

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ON THE STABILITY OF DIFFERENCE SCHEMES

It has repeatedly been conjectured that if a difference scheme is stable in the class of constant coefficients, then it is also stable in the class of variable coefficients. In this note an example is given showing that this conjecture is false if, as the class of variable coefficients, one takes piecewise-discontinuous and piecewise-differentiable functions.*

1. Consider the implicit difference scheme:

$$\frac{y_i^j - y_i^{j-1}}{\tau} = \frac{1}{h^2}\left[b_i\left(y_{i+1}^j - y_i^j\right) - a_i\left(y_i^j - y_{i-1}^j\right)\right] = L_h y_i^j, \quad 0 < i < N,\ j > 0, \tag{1}$$

and the first boundary-value problem corresponding to it:

$$y_0^j = y_N^j = 0, \quad j \geqslant 0; \tag{2}$$

$$y_i^0 = \varphi_i, \quad 0 < i < N, \tag{3}$$

where φ_i are prescribed initial values, and

$$a_i = k_i - \frac{1}{4}(k_{i+1} - k_{i-1}), \qquad b_i = k_i + \frac{1}{4}(k_{i+1} - k_{i-1}), \tag{4}$$

h and τ are the steps of the difference grid ($x_i = ih,\ t_j = j\tau,\ i = 0, 1, \dots, N;\ j = 0, \dots, m,\ h = 1/N,\ \tau = T/m$).

This scheme corresponds to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad 0 < x < 1,\ 0 < t \leq T,\ k(x) > 0, \tag{5}$$

as is easily verified by rewriting it in the form

$$\frac{y_i^j - y_i^{j-1}}{\tau} = k_i \frac{y_{i-1}^j - 2y_i^j + y_{i+1}^j}{h^2} + \frac{k_{i+1} - k_{i-1}}{2h} \frac{y_{i+1}^j - y_{i-1}^j}{2h}.$$

The difference scheme (1) with conditions (2) and (3) is stable in the class of continuous coefficients for sufficiently small h and arbitrary τ, since in this case $a_i > 0,\ b_i > 0$, and the maximum principle is valid, whence uniform correctness with respect to h follows.

2. We shall seek a solution of the difference equations (1) with conditions (2) in the form $y_i^j = s^j v_i$. For the function v_i we obtain the eigenvalue problem

$$L_h v_i = -\hat{\lambda} v_i, \quad 0 < i < N, \qquad v_0 = v_N = 0$$

or

$$L_h h v_i = \lambda v_i, \qquad v_0 = v_N = 0 \quad (\lambda = -\hat{\lambda}), \tag{6}$$

where $\lambda = (s-1)/s\tau$, so that $s = 1/(1-\lambda\tau)$.

Let $k(x) > 0$ be a piecewise-constant function

$$k(x) = \begin{cases} k_1, & \text{for } x < \xi = x_n + \theta h,\ 0 < \theta < 1, \\ k_2, & \text{for } x > \xi, \end{cases} \tag{7}$$

where ξ is the point of discontinuity of $k(x)$, irrational.

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* The main result of this work was presented in a report at the Fourth All-Union Mathematical Congress in 1961.

It will be shown that if

$$q_0 = \frac{a_n a_{n+1}}{\sqrt{k_1}} + \frac{b_n b_{n+1}}{\sqrt{k_2}} \tag{8}$$

is negative, then for sufficiently small $h < h_0$ problem (6) has a positive eigenvalue λ_0, with $\lambda_0 > z_0/h$, where z_0 is an arbitrary positive constant. It will then follow that, for $q_0 < 0$,

$$s^m = s^{T/\tau} \underset{\tau\to 0}{\longrightarrow} e^{\lambda_0 T} \geq e^{z_0 T/h} \qquad (m\tau = T).$$

For our scheme

$$q_0 = \frac{k_1^2}{\sqrt{k_2}} \left\{ \left[1 - \frac{1}{4}(\varkappa - 1)\right] \left[\varkappa - \frac{1}{4}(\varkappa - 1)\right] \sqrt{\varkappa} + \right.$$

$$\left. + \left[1 + \frac{1}{4}(\varkappa - 1)\right] \left[\varkappa + \frac{1}{4}(\varkappa - 1)\right] \right\},$$

where $\varkappa = k_2/k_1$.

If $k_2 > k_1$ ($\varkappa > 1$), then all square brackets except the first are positive; if $\varkappa > 5$, the first bracket is negative, and the first term inside the braces is also negative. Since the degree in $\varkappa$ of the first term is $5/2$, while that of the second is 2, it is clear that there exists a $\varkappa_0$ such that $q_0 < 0$ for $\varkappa > \varkappa_0$ (the approximate value is $\varkappa_0 \simeq 11.88\ldots$).

3. It is not difficult to verify that the function

$$v_i = \begin{cases} v_1 \operatorname{sh} \alpha x_i / \operatorname{sh} \alpha x_n, & \text{for } i \le n, \\ v_2 \operatorname{sh} \beta(1 - x_i) / \operatorname{sh} \beta(1 - x_{n+1}), & \text{for } i \ge n+1 \end{cases}$$

satisfies the conditions of problem (6) for $i < n$ and $i > n+1$, if the conditions

$$\lambda h^2 = 4k_1 \operatorname{sh}^2 \frac{\alpha h}{2} = 4k_2 \operatorname{sh}^2 \frac{\beta h}{2} \tag{9}$$

are satisfied.

The values v_1 and v_2 are determined from equations (6) for $i = n$ and $i = n+1$, and λ from the solvability condition for these equations with respect to v_1 and v_2.

Introduce the notation

$$\mu_1 = (\operatorname{sh} \alpha x_n - \operatorname{sh} \alpha x_{n-1}) / \operatorname{sh} \alpha x_n;$$

$$\mu_2 = [\operatorname{sh} \beta(1 - x_{n+1}) - \operatorname{sh} \beta(1 - x_{n+2})] / \operatorname{sh} \beta(1 - x_{n+1})$$

and write equations (6) for $i = n, n+1$ in the form

$$(\lambda h^2 + b_n + a_n \mu_1) v_1 - b_{nv} 2 = 0,$$

$$-a_{n+1} v_1 + (\lambda h^2 + a_{n+1} + b_{n+1} \mu_2) v_2 = 0.$$

Equating the determinant of this system to zero, we obtain an equation for λ:

$$F(\lambda, h) = \lambda^2 h^4 + p\lambda h^2 + q = 0,$$

where

$$p = b_n + a_{n+1} + a_n\mu_1 + b_{n+1}\mu_2,$$

$$q = a_{na_{n+1}}\mu_1 + b_{nb_{n+1}}\mu_2 + a_{nb_{n+1}}\mu_1\mu_2. \tag{10}$$

It is required to prove that, when the condition $q_0 < 0$ is fulfilled, for any z_0 there exists λ_0, a root of the equation $F(\lambda, h) = 0$, with $\lambda_0 > z_0/h$.

4. It is not difficult to see that $0 < \mu_1 < 1$, $0 < \mu_2 < 1$. Hence p and q are bounded for given k_1 and k_2. Consider arbitrary numbers z_1 and z_0 such that $z_1 < z_0$. Replace λ by z, putting $z = \lambda h$; then the function $F(\lambda, h)$ is transformed into the function

$$\overline{F}(z, h) = F(\lambda, h) = z^2h^2 + pzh + q.$$

In Sec. 5 it will be shown that

$$q = q_0\sqrt{zh}\,(1 + O(h)) + O(h), \tag{11}$$

where q_0 is determined by formula (8). If $q_0 < 0$ ($\varkappa > \varkappa_0$), then, for sufficiently small $h < h_0$,

$$\overline{F}(z, h) < 0 \quad \text{for } z_1 \leq z \leq z_0.$$

On the other hand, for any h, by virtue of the boundedness of p, q,

$$\overline{F}(z, h) > 0 \quad \text{for } z > Z(h) > z_0,$$

whence it follows that for $q_0 < 0$ there is a root of the equation $\overline{F}(z, h) = 0$ satisfying the condition $z > z_0$, or a root of the equation $F(\lambda, h) = 0$, for which $\lambda > z/h$, which proves the instability of scheme (1) for $\varkappa > \varkappa_0$.

5. Let us prove the asymptotic equality (11). It is obvious that, for the values of α and β determined by the equalities

$$zh = 4k_1 \operatorname{sh}^2\frac{\alpha h}{2} = 4k_2 \operatorname{sh}^2\frac{\beta h}{2} \tag{9'}$$

in the interval $z_1 \leq z \leq z_0$ the following asymptotic equalities in h hold:

$$\alpha h = \sqrt{\frac{zh}{k_1}}\,(1 + O(h)), \qquad \beta h = \sqrt{\frac{zh}{k_2}}\,(1 + O(h)). \tag{12}$$

Moreover, for $z_1 \le z \le z_0$,

$$\mu_1 = 2\,\mathrm{sh}\,\frac{\alpha h}{2}\,\mathrm{ch}\,\alpha(x_n - 0.5h)/\,\mathrm{sh}\,\alpha x_n =$$

$$= \alpha h\,[1 + O(\alpha^2 h^2)] = \alpha h\,[1 + O(h)] \tag{13}$$

and, analogously,

$$\mu_2 = \beta h\,(1 + O(h)). \tag{13'}$$

Substituting equalities (13), (13′) into expression (10) for q, we obtain

$$q = \left(\frac{a_{na_{n+1}}}{\sqrt{k_1}} + \frac{b_{nb_{n+1}}}{\sqrt{k_2}}\right)\sqrt{zh}\,(1 + O(h)) + O(h) = q_0\sqrt{zh}\,(1 + O(h)) + O(h),$$

where q_0 is determined by formula (8).

Thus the instability of scheme (1) for $\varkappa = k_2/k_1 > \varkappa_0$ and sufficiently small $h < h_0$ has been proved.

It is not difficult to see (cf. (1′)) that, in those cases in which scheme (7) for piecewise constant coefficients $k(x)$ converges, the limiting function for y_i^j will differ from the solution of the corresponding boundary-value problem for the differential equation (5). For convergence in the class of discontinuous coefficients, the scheme

$$\frac{y_i^j - y_i^{j-1}}{\tau} = L_{hy}i^j$$

must necessarily and sufficiently have the operator $L_{hy}i$ be conservative (self-adjoint), i.e., one must have $b_i = a_{i+1}$ (see (1′)).

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REFERENCES

1. A. N. Tikhonov, A. A. Samarskii, *Zhurn. vychisl. matem. i matem. fiz.*, 1, No. 1, 5 (1961).

Note: Figure translations are in progress. See original paper for figures.

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